(Exceptional) Generalised Geometry for Superstring Theory and M-Theory

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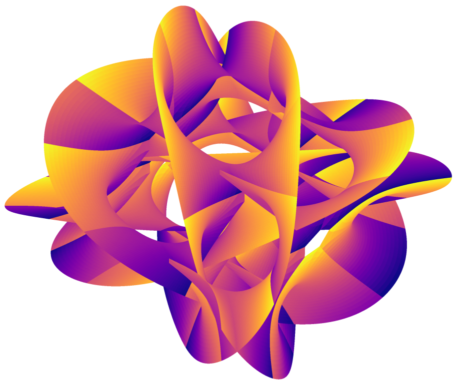

Abstract

This thesis is concerned with Hitchin's generalised geometry and its applications in superstring theory and, in particular, in its low-energy limit, supergravity. We provide a brief overview of the main mathematical structures that are spawned by the extension of the tangent bundle over a manifold by the cotangent bundle. We see that this generalisation results in a mathematical formalism whereby the backgrounds of ten-dimensional supergravity and their symmetries can be neatly embedded in the geometry, therefore offering a very natural description in terms of this generalised framework. Motivated by the elegance of this formalism, we review an extension that geometrises the remaining degrees of freedom of M-theory and type II supergravity. By building covariance under the larger U-duality group of string theory, we explore the description of type II and M-theory geometries in terms of exceptional generalised geometry, and the integrability of exceptional complex structures in supersymmetric compactifications of eleven-dimensional supergravity.

To nonno Pino.

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Chapter 1

Introduction

The objective of this dissertation is to describe a geometrical framework within which many facets of string theory and supergravity find a natural and elegant description: generalised geometry, and its generalisation to exceptional geometries.

Many of humanity's most profound ideas – from Newton's mechanics to Einstein's general relativity – have been developed at the interface between mathematics and physics. Like a well-choreographed dance, new discoveries and conjectures in physics lead to the birth of new areas of mathematics and to a refinement of existing ones; conversely, the conception of new mathematical ideas and structures, regardless of how abstract they might initially appear, can lead to a deeper understanding and appreciation of physical theories, or even to full paradigm shifts in the way we interpret and describe reality.

String theory has revealed itself to be a fertile ground for the common development of physics and mathematics, and Hitchin's generalised geometry [1–4] is, in some sense, only one of the latest steps in this eternal dance.

More concretely, we begin in chapter 2 of this thesis by introducing a number of notions from (ordinary) complex differential geometry. In particular, we introduce various G-structures whose generalised counterparts play important roles in the later sections of the thesis, as well as two geometries of fundamental importance: Kähler and Calabi-Yau geometries.

In chapter 3, we present an overview of Hitchin's generalised geometry; this is a formalism whereby the tangent bundle T over a d-dimensional manifold M is extended to the sum $T \oplus T^*$ of the tangent and cotangent bundles. This new space is endowed with a natural action of $\mathrm{SO}(d,d)$, whose discrete version is the T-duality symmetry group. We give a brief description of the geometry of this new bundle and present a generalisation of some of the mathematical structures discussed in the previous chapter, including metrics, brackets, G-structures and their integrability, geometries, and spinors. Particular emphasis is given on the integrability of the structures presented, as this will turn out to be related to the description of supersymmetric backgrounds. Furthermore, we discuss a particular patching which allows the geometry to fully capture the gerbe structure of the gauge fields. Ultimately, the objective of chapter 3 is to set up the mathematical stage for the string theory discussions of the following two chapters.

We open chapter 4 by introducing string theory compactifications. We can then begin to embed various notions from string theory and supergravity into the formalism of generalised geometry. For instance, we find that the metric and the B-field naturally combine into a generalised metric. The perturbative charges of the string – namely, its momentum and winding number – can be assembled into an $\mathrm{SO}(d,d)$ vector. After introducing T-dualities, together with the Buscher procedure, we review how the parameters associated with the symmetries of the Neveu-Schwarz-Neveu-Schwarz sector of string theory – namely, diffeomorphisms and B-gauge transformations – naturally inhabit the generalised tangent space. Returning to type II supergravity compactifications, we show how the imposition of supersymmetry leads to equations of motion whose implications on the geometry of the target space are best described in terms of generalised structures. We first explore the fluxless case, and then sequentially turn on Neveu-Schwarz-Neveu-Schwarz and Ramond-Ramond fluxes.

In type II string theory, $\mathrm{SO}(d,d)$ is only a small part of the group of U-dualities, conjectured by Hull and Townsend to be the unified symmetry group of string theory [5]. This motivates a generalisation of Hitchin's generalised geometry, with the exceptional group E_{d+1} taking the place of $\mathrm{SO}(d,d)$. This extension is reviewed in chapter 5, and it endows the formalism with a natural action on the gauge fields of the Ramond-Ramond sector. In particular, we review type IIA and M-theory geometries, and characterise how the former arise as a reduction of the latter. Following in the footsteps of the previous chapters, we describe how a geometrisation of the gauge symmetries of eleven-dimensional supergravity can be achieved by adorn-ing the exceptional generalised tangent bundle with a connective structure. We conclude by sketching how, in analogy with ordinary geometry, the compactifications of supersymmetric flux backgrounds of eleven-dimensional supergravity can be described in terms of exceptional complex structures [6, 7].

Chapter 2

Complex differential geometry

In this chapter, we provide a brief and incomplete overview of the facets of (complex) differential geometry that reappear, perhaps in a generalised form, in the subsequent chapters. In the interest of time, however, we assume some extent of familiarity with fibre bundle theory; for a concise and foundational review, see for instance [8].

2.1 G-structures

Let us begin by laying out the basic notation that we will employ in this thesis. We denote the tangent and cotangent bundles over a base manifold M as T and T^*, respectively. The space of sections of a bundle E is $\Gamma(E)$, so that, for instance, a vector field is an element of $\Gamma(T)$, while an r-form field is an element of $\Gamma(\Lambda^r T^*)$.

DEFINITION. A ***G*-structure**, where $G \subseteq \mathrm{GL}(d)$, is a principal G-subbundle $P_G \to M$ of the frame bundle[1] F, i.e. $P_G \subseteq F$.

The presence of a G-structure on a d-dimensional Riemannian manifold implies a reduction of the structure group of the frame bundle from $\mathrm{O}(d)$ to $G \subset \mathrm{O}(d)$ [9].

[1] Recall that a frambundle $F \to M$ is the disjoint union

$$F = \coprod_{p \in M} \{p, F_p\}$$

of the set F_p of all frames at a point $p \in M$.

Here, we will adopt the more practical working definition[2] that a G-structure is given by a set $\{\tau_i\}$ of globally non-vanishing tensors τ_i which are invariant under the action of a Lie group G, i.e.

$$\cap_i \operatorname{stab}(\tau_i) = G,$$

where $\operatorname{stab}(\tau_i) = \{g \in \mathrm{GL}(d) : g \cdot \tau_i = \tau_i\}$, and we used the natural action of the general linear group on a tensor.

For instance, a Riemannian metric $g \in \Gamma(S^2 T^*)$ defines a notion of orthogonality; $\mathrm{O}(d)$ is the group that preserves orthogonality, and so a metric defines an $\mathrm{O}(d)$ structure. A volume form $\mathrm{vol} \in \Gamma(\Lambda^d T^*)$, on the other hand, introduces a notion of scale on the manifold. For any $g \in \mathrm{GL}(d)$, we have that

$$g \cdot \mathrm{vol} = \det(g)\mathrm{vol},$$

and so vol defines an $\mathrm{SL}(n)$ structure. We will present more relevant examples – namely, complex, symplectic, Kähler, and Calabi-Yau structures – in the next few sections.

The definition above is a purely algebraic statement. We now explore the possibility of attaching differential conditions to G-structures.

Definition. A connection ∇ on T is **compatible** with a G-structure defined by a set $\{\tau_i\}$ of tensors if [10]

$$\nabla \tau_i = 0.$$

A rationale behind the study of compatible connections is that they respect the decomposition of a tensor bundle $E = R_1 \oplus R_2 \oplus \ldots$ into representations of G, for if $\lambda \in \Gamma(R_1)$, where the R_i are G-submodules, then $\nabla\lambda \in \Gamma(R_1)$ as well [11].

Compatible connections always exist [11]. A stronger constraint is the following.

Definition. A G-structure is **integrable** if there exists a torsion-free compat-

[2]Note that not all G-structures can be described in this way [9].

ible connection.

As we will see, this last statement translates into (a set of) differential constraints on the defining tensor(s) τ_i. This can be easily seen in a simple example where we take the τ_i's to be 1-forms with components $\omega_\mu^{(i)}$ in a coordinate basis; we then have, in the same basis,

$$\partial_{[\mu}\omega_{\nu]}^{(i)} = \nabla_{[\mu}\omega_{\nu]}^{(i)} + \Gamma^{\lambda}_{[\mu\nu]}\omega_{\lambda}^{(i)}.$$

For a compatible connection, $\nabla_\mu\omega_\nu^{(i)} = 0$. Furthermore, in the absence of torsion, $\Gamma^{\lambda}_{[\mu\nu]} = 0$. We thus find a differential constraint of the form $\partial_{[\mu}\omega_{\nu]}^{(i)} = 0$.

Later, we will find differential conditions relating to the integrability of various structures emerge in the form of the Killing spinor equations arising in supersymmetric compactifications.

2.2 Almost complex and symplectic structures

In the following, we consider a manifold M with $\dim_{\mathbb{R}}(M) = d$, where d is even, and use indices $i, j \in \{1, \ldots, d\}$.

Definition. An **almost complex structure** $J \in \mathrm{End}(T)$ is a (real) endomorphism of the tangent bundle T given by

$$J: \quad T \to T \quad \text{such that} \quad J^2 = J \circ J = -\mathrm{id}_T, \tag{2.1}$$

where id_T is the identity map on T.

The defining property $J^2 = -\mathrm{id}_T$ implies that, upon being equipped with an almost complex structure J, a (complexified) tangent bundle is partitioned by the action of J into two subbundles $T^{1,0} = P_- T^{1,0}$ and $T^{0,1} = P_+ T^{0,1}$, corresponding respectively to the $\pm i$ eigenspaces[3] of J:

$$T \otimes \mathbb{C} = T^{0,1} \oplus T^{1,0} = P_+ T^{0,1} \oplus P_- T^{1,0}, \tag{2.2}$$

[3]So that, for instance, a $(1,0)$-vector field with components X^i satisfies $J^j{}_i X^i = iX^j$.

where we introduced the projection operators[4]

$$P_\pm = \frac{1}{2}(\mathbb{I} \pm iJ), \tag{2.3}$$

which project down to $T^{0,1}$ and $T^{1,0}$, respectively [12, 13].

The (complexified) cotangent bundle splits in an analogous fashion under the action of J:

$$T^* \otimes \mathbb{C} = T^{*0,1} \oplus T^{*1,0}, \tag{2.4}$$

whereby $\alpha \in \Gamma(T^* \otimes \mathbb{C})$ is also a section of $T^{*0,1}$ if and only if $\alpha(X) = 0$ for all vector fields $X \in \Gamma(T^{1,0})$, and conversely for $T^{*1,0}$.

The integrability of an almost complex structure J is equivalent[5] to the involutility of the eigenbundle $T^{1,0}$ (or to that of $T^{0,1}$) under the Lie bracket:

$$P_\pm[P_\mp X, P_\mp Y] = 0 \quad \forall\ X, Y \in \Gamma(T), \tag{2.5}$$

meaning that the Lie bracket of two sections of either eigenbundle is also a section of that same eigenbundle.

The integrability of J can also be recast into the vanishing of the Nijenhuis tensor $N_J \in \Gamma(T \oplus \Lambda^2 T^*)$,

$$N_J(X, Y) = J[JX, Y] + J[X, JY] - [JX, JY] + [X, Y].$$

This is one of the forms in which the Newlander-Nirenberg theorem is sometimes stated [14].

An integrable almost complex structure is referred to simply as a complex structure. Its integrability implies that the local complex coordinates $\{z^i\}$ may be integrated, in the sense that the local one-forms $\{dz^i\}$ are truly their differentials [15].

The introduction of an almost complex structure reduces the structure group from $\mathrm{GL}(d;\mathbb{R})$ to $\mathrm{GL}(d/2;\mathbb{C})$. In terms of the language of G-structures introduced earlier, an almost complex structure is a $\mathrm{GL}(d/2;\mathbb{C})$ structure. We see that a topo-

[4] We can indeed show that $P_\pm^2 = P_\pm$, as expected of legitimate projection operators, as well as $P_+ + P_- = 1$.

[5] This equivalence is spelt out in Frobenius' theorem for distributions − technically, the subbundles $T^{1,0}$ and $T^{0,1}$ are distributions [12].

logical obstruction to having such a structure is the requirement for the manifold to be even-dimensional. At the level of representations, and for the case $d = 6$, the split in eq. (2.2) corresponds to the decomposition

$$\mathbf{6} \to \mathbf{3} \oplus \bar{\mathbf{3}}$$

of the fundamental representation, which is that in which the tangent bundle transforms.

The decomposition of the complexified cotangent bundle brought about by an almost complex structure (see eq. (2.4)) stimulates the following, more general decomposition for higher-degree forms [8],

$$\begin{aligned}\Lambda^n T^* \otimes \mathbb{C} &= \Lambda^n(T^{*1,0} \oplus T^{*0,1}) \\ &= \bigoplus_{p+q=n} (\Lambda^p T^{*1,0} \otimes \Lambda^q T^{*0,1}) \\ &= \bigoplus_{p+q=n} \Lambda^{p,q} T^*. \end{aligned} \tag{2.6}$$

This allows us to build a local section of $\Lambda^{d/2,0}T^*$ out of the local frame $\{\theta^a\}$ consisting of $d/2$ independent $(1,0)$-forms θ^a,

$$\Omega = \bigwedge_{k=1}^{d/2} \theta^k.$$

A form which can locally be written in this fashion is referred to as a decomposable form [12]. In turn, such a $d/2$-form Ω (which we emphasise need not be globally defined – Ω can be rescaled by a complex factor via a $\mathrm{GL}(d/2;\mathbb{C})$ transformation across patches) can be used to define a subbundle $T^{0,1}$ as a kernel, and so an almost complex structure J; the explicit construction is

$$T^{0,1} = \{V \in \Gamma(T) \mid \imath_V \Omega = 0\}. \tag{2.7}$$

We see then that the invariant form associated to an almost complex structure is a decomposable, complex $d/2$-form. We can now translate the integrability of an almost complex structure into a condition on Ω. Given vector fields $X, Y \in \Gamma(T^{0,1})$, from eq. (2.7) we have that

$$\imath_X \Omega = \imath_Y \Omega = 0 \quad \Rightarrow \quad \imath_Y \imath_X d\Omega = 0,$$

where we used

$$\imath_{[X,Y]}\Omega = \imath_Y \imath_X d\Omega.$$

In the language of forms, therefore, the integrability of a complex structure translates into

$$d\Omega = \xi \wedge \Omega \tag{2.8}$$

for some one-form ξ. When $\xi = 0$, Ω is closed, and we will see that this implies the holomorphical triviliaty of the canonical bundle. In the presence of a symplectic structure, this will lead to a Calabi-Yau manifold [12, 15].

2.3 Dolbeault operators and cohomology

The decomposition in eq. (2.6) allows us to refine the grading of forms; specifically, it develops the notion of n-forms into that of (p, q)-forms. In particular,

Definition. A complex n-form $\omega \in \Gamma(\Lambda^n T^*) \otimes \mathbb{C}$ is a $(\mathbf{p}, \mathbf{q})$**-form**, where $p + q = n$, if

$$\omega(V_1, \dots, V_n) \neq 0,$$

where $V_i \in \Gamma(T) \otimes \mathbb{C}$ for $i \in \{1, \dots, n\}$, only if p of the $V_i \in \Gamma(T^{1,0})$ and the remaining q of the $V_i \in \Gamma(T^{0,1})$ [8].

We label the space of (p, q)-form fields as $\Gamma(\Lambda^{p,q} T^*)$.

Now, given a (p, q)-form field η on a complex manifold,

$$d\eta \in \Gamma(\Lambda^{p+1,q} T^* \oplus \Lambda^{p,q+1} T^*)$$

so that we can slice the de Rham differential in two:

$$d = \partial + \bar{\partial} \tag{2.9}$$

where

$$\partial: \qquad \Gamma(\Lambda^{p,q}T^*) \to \Gamma(\Lambda^{p+1,q}T^*)$$

increases p by one, while

$$\bar{\partial}: \qquad \Gamma(\Lambda^{p,q}T^*) \to \Gamma(\Lambda^{p,q+1}T^*)$$

increases q by one. The operators ∂ and $\bar{\partial}$ are known as Dolbeault operators [8]. The nilpotency of the de Rham operator, $d^2 = 0$, then translates into

$$d^2 = \underbrace{\partial^2} + \underbrace{\partial\bar{\partial} + \bar{\partial}\partial} + \underbrace{\bar{\partial}^2} = 0,$$

where the three underlined operators are linearly independent from each other, and must therefore vanish individually. The nilpotency of the Dolbeault operators advocates the definition of the (p,q)-th Dolbeault cohomology group

$$H^{p,q}_{\overline{\partial}} = \frac{\{\eta \in \Gamma(\Lambda^{p,q}T^*) \mid \overline{\partial}\eta = 0\}}{\{\eta \sim \eta + \overline{\partial}\lambda \mid \lambda \in \Gamma(\Lambda^{p,q-1}T^*)\}},$$

the complex vector space consisting of equivalence classes of $\bar{\partial}$-closed (p,q)-forms differing by up to a $\bar{\partial}$-exact (p,q)-form [8].

We can decompose[6] the de Rham cohomology groups H^n_d into Dolbeault ones,

$$H^n_d = \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}. \tag{2.10}$$

We can expand the notion of a Hodge star $\star$ by defining $\bar{\star}\omega \equiv \star\bar{\omega}$, so that, on a d-dimensional manifold, $\bar{\star}$ sends (p,q)-forms to $(d-p, d-q)$-forms. We then have a symmetric inner product

$$\begin{aligned} (\,\cdot\,,\,\cdot\,): \qquad & \Gamma(\Lambda^{p,q}T^*) \times \Gamma(\Lambda^{p,q}T^*) \to \mathbb{R} \\ & \alpha, \beta \mapsto (\alpha, \beta) \equiv \int_M \alpha \wedge \bar{\star}\beta. \end{aligned}$$

[6] In fact, this decomposition is only possible if the $\partial\overline{\partial}$-lemma

$$\text{Im } \partial \cap \text{Ker } \overline{\partial} = \text{Im } \overline{\partial} \cap \text{Ker } \partial = \text{Im } \partial\overline{\partial}$$

holds [1, 12].

We can now implicitly define the operators $\partial^\dagger$ and $\bar{\partial}^\dagger$ by

$$(\alpha, \partial\beta) = (\partial^\dagger\alpha, \beta) \qquad \text{and} \qquad (\alpha, \bar{\partial}\beta) = (\bar{\partial}^\dagger\alpha, \beta),$$

which are adjoint to the Dolbeault operators with respect to the above inner product. We can thus devise the following three Laplacians:

$$\begin{aligned} \nabla \ &= dd^\dagger + d^\dagger d \ = (d + d^\dagger)^2, \\ \nabla_\partial &= \partial\partial^\dagger + \partial^\dagger\partial = (\partial + \partial^\dagger)^2, \\ \text{and} \qquad \nabla_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial} = (\overline{\partial} + \overline{\partial}^\dagger)^2. \end{aligned}$$

In complete analogy with Hodge theory, we define ∂- and $\overline{\partial}$-harmonic (p, q)-forms α and β, respectively, via $\nabla_\partial\alpha = 0$ and $\nabla_{\overline{\partial}}\beta = 0$. We label the spaces of such forms as $\mathrm{Harm}^{p,q}_{\partial}(M)$ and $\mathrm{Harm}^{p,q}_{\overline{\partial}}(M)$.

The complexification of Hodge's decomposition theorem[7] proclaims the uniqueness of the following decomposition into orthogonal spaces:

$$\Gamma(\Lambda^{p,q}T^*) = B^{p,q}_{\overline{\partial}}(M) \oplus B^{\dagger p,q}_{\overline{\partial}}(M) \oplus \mathrm{Harm}^{p,q}_{\overline{\partial}}(M),$$

where $B^{p,q}_{\overline{\partial}}(M) = \overline{\partial}\Gamma(\Lambda^{p,q-1}T^*)$ is the space of $\overline{\partial}$-closed (p, q)-forms, and $B^{\dagger p,q}_{\overline{\partial}}(M) = \overline{\partial}^\dagger\Gamma(\Lambda^{p,q+1}T^*)$ is the space of $\overline{\partial}$-co-closed (p, q)-forms (i.e. $\overline{\partial}^\dagger\alpha = 0$) [8].

On a complex, d-dimensional manifold M we can define the $(d + 1)^2$ Hodge

[7]Recall that this states the existence of a unique decomposition of an r-form into a closed, a co-closed, and a harmonic r-forms on a compact, orientable Riemannian manifold without boundary.

numbers $h^{p,q} = \mathrm{dim} H^{p,q}_{\bar{\partial}}(M)$; these can be arranged into a Hodge diamond,

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 \iddots & & & & & & \ddots \\
h^{d,0} & h^{d-1,1} & & \vdots & & h^{1,d-1} & h^{0,d} \\
 \ddots & & & & & & \iddots \\
 & h^{d,d-2} & & h^{d-1,d-1} & & h^{d-2,d} & \\
 & & h^{d,d-1} & & h^{d-1,d} & & \\
 & & & h^{d,d} & & &
\end{array} \quad . \tag{2.11}
$$

Recall that the number of linearly independent n-forms, i.e. the dimension of $\mathrm{Harm}^n(M)$, is a topological invariant[8], known as the Betti number b^n. In general, Hodge numbers are not topological invariants[9]; however, they play an important role in discussions on effective theories in string compactifications, as we shall mention later.

2.4 Kähler and Calabi-Yau geometries

We now introduce a second structure which we will encounter frequently in later discussions.

[8] This is related to the existence of index theorems stating that the number of solutions of a differential equation depends only on the topology of the manifold.

[9] Nonetheless, certain linear combinations of Hodge numbers are topological invariants. These were classified by Kotschic and Schreieder in [16].

DEFINITION. A **pre-symplectic structure** $\omega \in \Gamma(\Lambda^2 T^*)$ satisfies

$$\omega^{d/2} = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{d/2 \text{ times}} \neq 0,$$

so that it is non-degenerate.

In the language of G structures, ω is an $\mathrm{Sp}(d;\mathbb{R})$ structure. The integrability of a pre-symplectic structure consists in the closure of ω, i.e.

$$d\omega = 0.$$

We refer to integrable pre-symplectic structures simply as symplectic structures[10].

Consider now complexifying the tangent space $T_P M$ at a point p on the manifold M to give $T_p M \otimes \mathbb{C}$. We can extend a linear operator A to act on this complexified space as [8]

$$A(X + iY) = A(X) + iA(Y),$$

for $X, Y \in T_p M$. Similarly, by denoting $Z = X + iY \in T_p M \otimes \mathbb{C}$ and $W = U + iV \in T_p M \otimes \mathbb{C}$, we extend the metric at a point as

$$g_p(Z, W) = g_p(X, U) - g_p(Y, V) + i(g_p(X, V) + g_p(Y, U)).$$

DEFINITION. A complex manifold M with a complex structure[11] J is **Hermitian** if it has a metric g such that

$$g_p(J_p X, J_p Y) = g_p(X, Y) \tag{2.12}$$

for all $X, Y \in T_p M$, and at all points $p \in M$.

[10] Note that the terminology used in this section is not universal. Here, we adopt that of [12], while [17], for instance, uses the closure constraint, rather than the non-degeneracy one, to define pre-symplectic structures.

[11] If J is an *almost* complex structure, then M is an *almost* Hermitian manifold.

In a complex basis[12] $\{z^\mu\}$, a Hermitian metric only has mixed components, i.e. the purely holomorphic $g_{\mu\nu}$ or antiholomorphic $g_{\overline{\mu}\overline{\nu}}$ components vanish, while $g_{\overline{\mu}\nu}$ and $g_{\mu\overline{\nu}}$ in general do not [8]. For instance, on a manifold with $\mathrm{U}(n)$ holonomy, the fundamental vector of $\mathrm{SO}(2n)$ decomposes into the modules $\mathbf{n} \oplus \overline{\mathbf{n}}$ of $\mathrm{U}(n)$, and the fact that $g_{\overline{\mu}\nu} = g_{\mu\overline{\nu}} = 0$ reflects the absence of $\mathrm{U}(n)$ singlets from $\mathbf{n} \otimes \mathbf{n}$ and $\overline{\mathbf{n}} \otimes \overline{\mathbf{n}}$ for $n > 2$ [18].

Note that the action of J on X returns a vector that is orthogonal to X with respect to the Hermitian metric:

$$g_p(J_pX, X) = g_p(J_p^2X, J_pX) = -g_p(X, J_pX),$$

from which $g_p(J_pX, X) = 0$ follows.

Furthermore, we note that any complex manifold admits a Hermitian metric – indeed, the above definition should be interpreted as a constraint on the metric, rather than on the manifold itself [19]. Given a general Riemannian metric h, a Hermitian metric can always be built via

$$g_p(X, Y) = \frac{1}{2}(h_p(X, Y) + h_p(J_pX, J_pY)).$$

In components, eq. (2.12) implies

$$g_{ij} J^i{}_k J^j{}_\ell = g_{k\ell}. \tag{2.13}$$

It follows that

$$\omega_{ij} \equiv g_{ik} J^k{}_j \tag{2.14}$$

are the components of a pre-symplectic 2-form, referred to as the Kähler form [8];

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[12] If

$$\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\}$$

spans $T_P M$, then

$$\left\{ \frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right), \frac{\partial}{\partial \overline{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right) \right\}$$

is a basis for $T_P M \otimes \mathbb{C}$, and similarly for $T_P^* M \otimes \mathbb{C}$ [8].

to see this, we contract eq. (2.13) with $J^{\ell}{}_{m}$ to find

$$\omega_{km} = J^{j}{}_{\ell} J^{\ell}{}_{m} \omega_{jk} = -\omega_{mk},$$

where we used $J^{j}{}_{\ell} J^{\ell}{}_{m} = -\delta^{j}_{m}$. The Hermiticity condition in eq. (2.13) can thus be understood as a compatibility condition between g and J to define a 2-form ω.

We can also provide a coordinate-free definition of ω as

$$\omega(X, Y) = g(X, JY), \tag{2.15}$$

from which antisymmetry and invariance under J follow trivially.

Note that any two of $\{g, \omega, J\}$ determine the third – for instance, substituting eq. (2.14) into eq. (2.13), we find

$$g_{ij} = -\omega_{ik} J^{k}{}_{j}, \tag{2.16}$$

with the compatibility condition between ω and J being

$$\omega_{jk} J^{j}{}_{\ell} J^{k}{}_{i} = \omega_{\ell i},$$

which may be found by substituting for g_{ij} in eq. (2.13) and contracting with $J^{k}{}_{i}$.

In a complex basis, we may write the natural two-form as

$$\omega = -i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\bar{\nu}}, \tag{2.17}$$

since $\omega_{\mu\bar{\nu}} = g_{\mu\bar{\lambda}} J^{\bar{\lambda}}{}_{\bar{\nu}} = -i g_{\mu\bar{\nu}}$, where $J^{\bar{\mu}}{}_{\bar{\nu}} = -i\delta^{\bar{\mu}}_{\bar{\nu}}$. We see that the 2-form ω is in fact a (1,1)-form. We can also check it is real:

$$\overline{\omega} = i\overline{g_{\mu\bar{\nu}}} d\bar{z}^{\bar{\mu}} \wedge dz^{\nu} = -i g_{\nu\bar{\mu}} dz^{\nu} \wedge d\bar{z}^{\bar{\mu}} = \omega,$$

where we used $\overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}}$ and the antisymmetry of the wedge product.

We can rephrase the definition above into the language of G-structures.

Definition. A **Hermitian structure** on (the frame bundle of) an oriented smooth manifold is a Riemannian structure (given by the metric g) together with a complex structure (given by J), such that ω in eq. (2.15) is a non-degenerate 2-form.

The fact that a Hermitian structure is defined by a pre-symplectic structure and a complex one implies that it is a $\mathrm{Sp}(d;\mathbb{R}) \cap \mathrm{GL}(d/2;\mathbb{C}) = \mathrm{U}(d/2)$ structure.

We conclude by commenting that, for Hermitian geometries, it is useful to consider the (unique) metric-compatible Hermitian connection $\tilde{\nabla}$ with connection coefficients

$$\Gamma^{\mu}{}_{\rho\sigma} = g^{\overline{\nu}\mu} g_{\sigma\overline{\nu},\rho}, \tag{2.18}$$

for which the complex structure is parallel [8],

$$\tilde{\nabla} J = 0.$$

Definition. A Hermitian manifold is **Kähler** if its Kähler form satisfies the Kähler condition

$$d\omega = 0,$$

i.e. ω is a symplectic form.

We see that Kähler geometries lie at the intersection between complex and symplectic ones. We find,

$$\begin{aligned}
(\nabla_Z \omega)(X,Y) &= \nabla_Z[\omega(X,Y)] - \omega(\nabla_Z X, Y) - \Omega(X, \nabla_Z Y) \\
&= \nabla_Z[g(JX,Y)] - g(J\nabla_Z X, Y) - g(JX, \nabla_Z Y) \\
&= (\nabla_Z g)(JX,Y) + g(\nabla_Z JX, Y) - g(J\nabla_Z X, Y) \\
&= g(\nabla_Z JX - J\nabla_Z X, Y) = g((\nabla_Z J) X, Y),
\end{aligned}$$

where $\nabla\omega$ is the form with components $\nabla_i \omega_{jk}$, and ∇ is the Levi-Civita connection, so that $\nabla_Z g = 0$ and $\nabla\omega = d\omega$. The closure of ω then translates into $\nabla J = 0$, and vice versa – on a Kähler manifold, the Hermitian and Christoffel connections coincide [19].

The closure of ω allows for the following definition.

Definition. The **Kähler class** is the cohomology class $[\omega]$ of the Kähler form ω.

Using the split in eq. (2.9), the Kähler condition becomes

$$\partial\omega = \overline{\partial}\omega = 0,$$

and so locally we have that

$$\omega = \partial\omega = \overline{\partial}\alpha,$$

for some 1-forms α and ω; therefore,

$$\omega = -i\partial\overline{\partial}\mathcal{K}$$

for some 0-form $\mathcal{K}$, referred to as the Kähler potential [18]. From eq. (2.17), we then have that

$$g_{\alpha\overline{\beta}} = \partial_\alpha\partial_{\overline{\beta}}\,\mathcal{K},$$

so that the Kähler potential determines the geometry. This implies that, in a patch overlap $U_{(i)} \cap U_{(j)}$, the "gauge transformation"

$$\mathcal{K}_{(i)}(z,\overline{z}) = \mathcal{K}_{(i)}(z,\overline{z}) + f_{(i)}(z) + \overline{f}_{(i)}(\overline{z})$$

for a holomorphic functions f, leaves the geometry unchanged.

It can be shown that the only non-vanishing components of the connection are

$$\Gamma_{\mu\nu}{}^{\rho} = g^{\rho\overline{\sigma}}\partial_\mu g_{\nu\overline{\sigma}},$$

and those related by conjugation. The affine connection given by the components above implies covariant constancy of the complex structure J. Its stabiliser within $\mathrm{SO}(d)$ is $\mathrm{U}(d/2)$, so that metrics with $\mathrm{U}(d)$ holonomy are Kähler metrics [18].

It also follows that the curvature tensor has non-zero components $R_{\mu\overline{\nu}\rho\overline{\sigma}}$, such that

$$R^{\overline{\mu}}{}_{\overline{\nu}\rho\overline{\sigma}} = g^{\overline{\mu}\mu}R_{\mu\overline{\nu}\rho\overline{\sigma}} = \Gamma_{\overline{\nu}\overline{\sigma}}{}^{\overline{\mu}}{}_{,\rho}\,.$$

It follows that the Ricci tensor is Hermitian ($R_{\alpha\beta} = R_{\overline{\alpha}\overline{\beta}} = 0$), with components

$$R_{\overline{\mu}\nu} = -\Gamma_{\overline{\mu}\overline{\sigma}}^{\overline{\sigma}}{}_{,\nu} = -\partial_{\overline{\mu}}\partial_{\nu} \log \det g_{\gamma\overline{\delta}}.$$

From this, we can define a Ricci form

$$\mathfrak{R} = -iR_{\alpha\overline{\beta}}dz^{\alpha} \wedge d\overline{z}^{\overline{\beta}}$$

so that

$$\mathfrak{R} = i\partial\overline{\partial} \log \det g_{\gamma\overline{\delta}}. \tag{2.19}$$

Since $\partial\overline{\partial} = -d(\partial - \overline{\partial})/2$, it follows that $\mathfrak{R}$ is closed, $d\mathfrak{R} = 0$. We refer to the cohomology class $c_1(M) = [\mathfrak{R}/2\pi] \in H^2(M; \mathbb{R})$ as the first Chern class of the manifold [8].

DEFINITION. A **Calabi-Yau** manifold is a compact Kähler manifold with vanishing first Chern class,

$$c_1(M) = 0.$$

Calabi famously conjectured that, for a compact Kähler manifold and given a Kähler class $[\omega]$, there exists a unique Kähler metric whose corresponding Kähler form $\tilde{\omega}$ lies in $[\omega]$, and such that $\tilde{\omega}$'s Ricci form belongs in the manifold's first Chern class. Since the vanishing of the Ricci form is a restatement of Ricci flatness, it follows that Calabi-Yau manifolds, for which the first Chern class vanishes, admit a unique Ricci-flat metric for each Kähler class. The proof of the Calabi conjecture was eventually completed by Yau in [20].

Looking at eq. (2.19), the condition of Ricci flatness, $\mathfrak{R} = 0$, can be restated as

$$\log \det g_{\gamma\overline{\delta}} = f + \overline{f}. \tag{2.20}$$

We note that, under a holomorphic coordinate transformation $z \to z'(z)$,

$$g_{\alpha\overline{\beta}} \to g'_{\alpha\overline{\beta}} = g_{\gamma\overline{\delta}} \frac{\partial z^{\gamma}}{\partial z'^{\alpha}} \frac{\partial \overline{z}^{\overline{\delta}}}{\partial \overline{z}'^{\overline{\beta}}},$$

so that

$$\log\det g'_{\gamma\bar{\delta}} = \log\det g_{\gamma\bar{\delta}} + \log\det \frac{\partial z^{\delta}}{\partial z'^{\gamma}} + \log\det \frac{\partial \bar{z}^{\bar{\delta}}}{\partial \bar{z}'^{\bar{\gamma}}},$$

where the final two terms on the right-hand side are holomorphic and antiholomorphic, respectively, and can be used to cancel the f and $\bar{f}$ terms in eq. (2.20). Assuming that such a transformation has been performed, so that we can drop f and $\bar{f}$, we have

$$\log\det g_{\gamma\bar{\delta}} = 0,$$

and so

$$\det\left(\frac{\partial^2 \mathcal{K}}{\partial z^{\alpha}\partial \bar{z}^{\bar{\beta}}}\right) = 1,$$

which is known as the Monge-Ampère equation.

On a d-dimensional compact Kähler manifold, the vanishing of the first Chern class is equivalent to the existence of a globally defined, nowhere-vanishing holomorphic $d/2$-form Ω [19]. That is, a form such that

$$\Omega \wedge \bar{\Omega} = \mathrm{vol},$$

and that is decomposable,

$$\Omega = \frac{F}{(d/2)!}\epsilon_{\alpha_1\ldots\alpha_{d/2}} dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_{d/2}}$$

for some holomorphic function F. Combining these last two results, we find that

$$F\bar{F} = \sqrt{\det g_{\gamma\bar{\delta}}}.$$

Taking the logarithm of this recovers an equation of the same form as eq. (2.20); we may then transform to a coordinate system in which $F = 1$ and find a Monge-Ampère equation, so that $\mathfrak{R} = 0$ and therefore $c_1(M) = 0$.

We see that a Calabi-Yau manifold naturally comes with the real Kähler 2-form ω and a complex, non-degenerate and decomposable $d/2$-form Ω that are compatible with each other – in the sense that $\omega \wedge \Omega = 0$, and that the associated metric[13]

[13] In 6 dimensions, a real 3-form ρ satisfying certain stability conditions leads to both an almost complex structure, via

$$J^i{}_p = \pm H(\rho)^{-1}\varepsilon^{ijk\ell mn}\rho_{pjk}\rho_{\ell mn},$$

is positive definite. The former defines an $\mathrm{Sp}(d)$ structure, while the latter an $\mathrm{SL}(d;\mathbb{C})$ one. The structure group of Calabi-Yau manifolds therefore corresponds to the intersection $\mathrm{SU}(d/2)$.

The Hodge diamond (see eq. (2.11)) for Calabi-Yau manifolds is particularly simple. First of all, we notice that on a Kähler manifold, the closure of ω under the de Rham operator allows us to decompose the de Rham cohomology groups H^n_d into the Dolbeault ones $H^{p,q}_{\bar\partial}$, as in eq. (2.10). The presence of a metric, and its associated Hodge $\star$, leads to the isomorphisms

$$H^{p,q}_{\bar\partial} \simeq H^{d/2-p,d/2-q}_{\bar\partial}.$$

Furthermore, the presence of a closed Ω engenders the additional isomorphisms

$$H^{0,q}_{\bar\partial} \simeq H^{d/2,q}_{\bar\partial} \quad \text{and} \quad H^{p,0}_{\bar\partial} \simeq H^{p,d/2}_{\bar\partial}.$$

The considerations above lead to the following Hodge diamond for a six-dimensional Calabi-Yau manifold:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} ,$$

which is entirely parametrised by $h^{1,1}$ and $h^{2,1}$ [19].

When introducing effective field theories, we will employ bases

$$\{r_a\}, \quad \{\tilde r^a\}, \quad \text{and} \quad \{\alpha_K, \tilde\alpha^K\} \tag{2.21}$$

for $H^{1,1}_{\bar\partial}$, $H^{2,2}_{\bar\partial}$, and H^3_d, respectively, where α_K and $\tilde\alpha^K$ are real forms. We can choose these in such a way that they entertain the following "orthogonality" relations

and a complex, decomposable 3-form Ω with

$$\mathrm{Re}(\Omega) = \rho, \quad \mathrm{Im}(\Omega) = J^i{}_j(\imath_i \wedge dx^j - dx^j \wedge \imath_i)\rho/6,$$

where $H(\rho)$ is the Hitchin function; we can then form a metric via eq. (2.16) [12, 21].

[22, 23],

$$\int r_a \wedge \tilde{r}^b = \delta_a^b$$

and

$$\int \alpha_K \wedge \tilde{\alpha}^J = \delta_K^J.$$

Note that a basis for $H_{\bar{\partial}}^{0,0} \oplus H_{\bar{\partial}}^{1,1}$ is given simply by $\{r_0 \equiv 1, r_a\}$, while $\{\text{vol}, \tilde{r}^a\}$ is a basis for $H_{\bar{\partial}}^{3,3} \oplus H^{2,2}$.

2.5 Torsion classes for SU(3) structures

Consider a G-invariant form η. In general, the torsion tensor T takes values in

$$\Lambda^1 T^* \otimes \Lambda^2 T^*, \quad \text{where} \quad \Lambda^2 T^* \simeq \mathfrak{so}(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

$\mathfrak{g}^\perp$ being the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(d)$. We may then write the torsion tensor's components as $T_{mn}{}^p$, where the index p labels $\Lambda^1 T^*$, while m and n span $\Lambda^2 T^*$.

Consider now a connection ∇' for which $\nabla' \eta = 0$. Given the Levi-Civita connection ∇, we can drop $\mathfrak{g}$ when acting on G-invariant forms η, as in $\nabla \eta = (\nabla - \nabla')\eta$. This is in correspondence with what is referred to as the intrinsic torsion T_0; it has components in $\Lambda^1 T^* \otimes \mathfrak{g}^\perp$ [9]. The intrinsic torsion provides an obstruction to the integrability of the G-structure, since a compatible connection is torsion-free if and only if the G-structure has no intrinsic torsion [11].

We now specialise to the case of $G = \mathrm{SU}(3)$. This will turn out to be of physical relevance later. Observing the decomposition of $\nabla \eta$ into SU(3)-modules allows us to distinguish between different structures, as we will now see.

For the case of SU(3), and noting the decompositions in eq. (4.43), the space of intrinsic torsions decomposes into SU(3) modules as [23]

$$(\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}) = (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus 2(\mathbf{3} \oplus \bar{\mathbf{3}}).$$

We can therefore identify 5 tensors, the so-called torsion classes: a complex scalar

$$\mathcal{W}_1 \in \mathbf{1} \oplus \mathbf{1},$$

a complex primitive[14] (1,1)-form

$$\mathcal{W}_2 \in \mathbf{8} \oplus \mathbf{8},$$

a real primitive $(2,1)+(1,2)$ form

$$\mathcal{W}_3 \in \mathbf{6} \oplus \bar{\mathbf{6}},$$

a real one-form $\mathcal{W}_4$, and a complex $(1,0)$-form $\mathcal{W}_5$.

On an almost complex manifold, the exterior derivative of a (p,q)-form produces a $(p+2,q-1)$-form, a $(p+1,q)$-form, a $(p,q+1)$-form, and a $(p-1,q+2)$-form, with the first and last terms not appearing in the case of the manifold being complex. Therefore, given the $(1,1)$ Kähler form ω, it follows that $d\omega$ is made up of a $(3,0)$-form, a $(2,1)$-form, a $(1,2)$-form, and a $(0,3)$-form. The $(3,0)+(0,3)$ part transforms in

$$\mathbf{1} \oplus \mathbf{1},$$

while the $(2,1)$-form transforms in

$$\mathbf{6} \oplus \bar{\mathbf{3}},$$

so that the $(2,1)+(1,2)$ part in total transforms in

$$(\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus 2(\mathbf{3} \oplus \bar{\mathbf{3}}).$$

We conclude that $d\omega$ must contain the torsion classes $\mathcal{W}_1$, $\mathcal{W}_3$, and $\mathcal{W}_4$. A more careful analysis shows that

$$d\omega = -\frac{3}{2}\mathrm{Im}(\overline{\mathcal{W}}_1\Omega) + \mathcal{W}_4 \wedge \omega + \mathcal{W}_3, \tag{2.22}$$

with the primitivity condition taking the form [12]

$$\mathcal{W}_3 \wedge \omega = 0.$$

[14]By primitive we mean that it satisfies $\mathcal{W}_{2ij}J^{ij} = 0$.

Vanishing torsion classes					Geometry
$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_4$	$\mathcal{W}_5$	
■	■				complex
■		■	■		symplectic
■	■		■	■	special Hermitian
	■	■	■	■	nearly Kähler
■		■	■	■	almost Kähler
■	■	■	■		Kähler
■	■	■	■	■	Calabi-Yau

Table 2.1: Various geometries (final column) in terms of their defining vanishing classes, from $\mathcal{W}_1$ (leftmost column) to $\mathcal{W}_5$. Coloured [uncoloured] cells correspond to [non-]vanishing classes [23].

Similarly, it can be shown that, for the complex $(3,0)$-form Ω,

$$d\Omega = \mathcal{W}_1 \omega \wedge \omega + \mathcal{W}_2 \wedge \omega + \overline{\mathcal{W}}_5 \wedge \Omega, \tag{2.23}$$

with $\mathcal{W}_2$ primitive in the sense that

$$\mathcal{W}_2 \wedge \omega \wedge \omega = 0.$$

We can now provide a useful classification of various geometries in terms of their vanishing torsion classes. For instance, on a complex manifold, acting with d on a (p,q)-form does not spawn $(p+2, q-1)$- and $(p-1, q+2)$-forms, and so, by looking at eqs. (2.22) and (2.23), we conclude that[15]

$$\mathcal{W}_1 = \mathcal{W}_2 = 0.$$

Similarly, on a symplectic manifold $d\omega = 0$, and so from eq. (2.22) we expect

$$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0.$$

As mentioned, a Kähler manifold is both symplectic and complex, and so is charac-

[15]It can be shown that $\mathcal{W}_1 = \mathcal{W}_2 = 0$ is also a sufficient condition for the manifold to be complex [23].

terised by

$$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0.$$

Finally, on a Calabi-Yau both ω and Ω are closed, which implies

$$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0.$$

Table 2.1 summarises the classifications argued above, and also provides some additional examples.

We will later see that introducing fluxes in the context of string compactifications generates non-vanishing intrinsic torsion, with profound consequences in terms of the geometry of the internal space.

Chapter 3

Generalised geometry

The framework of generalised geometry is built on the replacement of the tangent bundle T with $T \oplus T^*$, which sees the tangent and cotangent bundles take part on an equal footing. This generalisation endows us with a framework within which the group $\mathrm{SO}(d, d)$ has a natural action. This formalism elegantly covariantises the symmetries of string theory, as we shall see in the next chapters.

3.1 Geometry of $T \oplus T^*$

A section $\mathbb{X}$ of $T \oplus T^*$ consists of a vector field $X \in \Gamma(T)$ and a one-form $\xi \in \Gamma(T^*)$. We may write such a "generalised vector field" as the formal sum

$$\mathbb{X} = X + \xi \in \Gamma(T \oplus T^*),$$

or equivalently, using a natural matrix notation,

$$\mathbb{X} = \begin{pmatrix} X \\ \xi \end{pmatrix} \in \Gamma(T \oplus T^*).$$

If the base manifold is d-dimensional, then the fibre of $T \oplus T^*$ is a $2d$-dimensional vector space. Furthermore, there exists a natural inner product between generalised vectors [4],

$$\langle \mathbb{X}, \mathbb{Y} \rangle = \frac{1}{2}(\imath_X \upsilon + \imath_Y \xi) = \mathbb{X}^T \mathcal{I} \mathbb{Y},$$

where $\mathbb{X} = X + \xi$ and $\mathbb{Y} = Y + \upsilon$ are sections of $T \oplus T^*$, and the natural pairing metric is

$$\mathcal{I} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.1}$$

This canonical fibre metric is maximally indefinite with split signature (d, d). Therefore, it defines an $\mathrm{O}(d, d)$ structure – the subgroup under whose action the canonical metric is preserved [4]. Generalised vectors $\mathbb{X}$ transform under the vector representation of $\mathrm{O}(d, d)$, i.e. $\mathbb{X} \to g\mathbb{X}$, where g satisfies $g^T \mathcal{I} g = \mathcal{I}$.

$T \oplus T^*$ is associated to a $\mathrm{GL}(d; \mathbb{R})$ principal bundle: its transition functions are in $\mathrm{GL}(d; \mathbb{R})$. This reflects the transition functions of the base manifold M being diffeomorphisms. Nevertheless, the existence of the canonical bilinear form above, as well as that of a canonical orientation, suggests that we regard $T \oplus T^*$ as having structure group $\mathrm{SO}(d, d)$ [4, 6]. We will take this view throughout the following discussion, and return to this subtlety in section 3.9.

The Lie algebra $\mathfrak{so}(d, d)$ of the structure group $\mathrm{SO}(d, d)$ is composed of elements of the form [6]

$$\begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix},$$

where $A \in \mathrm{End}(T)$, B is a map $T \to T^*$, and β is a map $T^* \to T$. Being skew, B and β are identified with a 2-form and a bivector, respectively, while A is a $d \times d$ matrix[16]. In particular, A generates matrices of the form

$$\begin{pmatrix} M & 0 \\ 0 & M^{-T} \end{pmatrix}, \tag{3.2}$$

where $M \in \mathrm{GL}(d; \mathbb{R})$. We see that this embeds the action of $\mathrm{GL}(d; \mathbb{R})$ into the structure group of $T \oplus T^*$. On the other hand, the 2-form B generates elements

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \tag{3.3}$$

[16]Indeed, this coincides with the decomposition

$$\begin{aligned} \mathfrak{so}(T \oplus T^*) &= \Lambda^2(T \oplus T^*) \\ &= \mathrm{End}(T) \oplus \Lambda^2 T^* \oplus \Lambda^2 T. \end{aligned}$$

which act on generalised vectors to give the so-called "B-transformations"

$$e^B : \qquad X + \xi \mapsto e^B(X+\xi) = X + (\xi + \imath_X B). \tag{3.4}$$

Similarly, the bivector β generates "β-transformations", which send

$$X + \xi \to (X + \imath_\xi \beta) + \xi.$$

In the following, we will mostly be interested in the geometric subgroup[17] of $\mathrm{SO}(d,d)$,

$$\mathrm{GDiff} \equiv \mathrm{GL}(d;\mathbb{R}) \ltimes \Omega^2_{\mathrm{cl}}, \tag{3.5}$$

generated by A and closed B above, where Ω^2_{cl} is the space of closed 2-forms [6]. We will occasionally refer to this as the generalised diffeomorphism group [12], as it will turn out to be the gauge symmetry group of the Neveu-Schwarz-Neveu-Schwarz sector of superstring theory.

3.2 Generalised metrics

We can further introduce a positive-definite, generalised (Riemannian) metric $\mathcal{G}$ on $T \oplus T^*$ satisfying[18] $\mathcal{G}^2 = \mathbb{1}$, meaning it is compatible with the natural metric $\mathcal{I}$ [4]. Its introduction therefore partitions the generalised tangent bundle into subbundles $C_\pm \subset T \oplus T^*$, corresponding respectively to the ± 1 eigenspaces of $\mathcal{G}$ [24]. Conversely, the specification of a subbundle C_+ on which $\mathcal{I}$ is positive-definite – and so automatically also the specification of its complement C_-, which is orthogonal to it with respect to the natural metric, in the sense that $\mathcal{I}(C_+, C_-) = 0$ – defines a positive-definite metric via

$$\mathcal{G}(\mathbb{X}, \mathbb{Y}) = \mathcal{I}(\mathbb{X}, \mathbb{Y})|_{C_+} - \mathcal{I}(\mathbb{X}, \mathbb{Y})|_{C_-} \tag{3.6}$$

for generalised vectors $\mathbb{X}$ and $\mathbb{Y}$ [4].

Reformulating the specification of a generalised metric into the definition of subbundles $C_\pm$ on which $\mathcal{I}$ is $\pm$-definite makes it evident that the structure group is

[17] A 2-form has $d(d-1)/2$ components in d dimensions.
[18] This makes $\mathcal{G}$ an almost local product structure [6].

reduced from $\mathrm{O}(d,d)$ to its maximal compact subgroup[19], $\mathrm{O}(d) \times \mathrm{O}(d)$. The latter is indeed the (largest) subgroup which separately preserves the restrictions of $\mathcal{I}$ to $C_{\pm}$.

Note that it is common in the more physics-oriented literature to define the generalised metric as the positive-definite metric $\mathcal{H}$ on $T \oplus T^*$ such that

$$\mathcal{I}^{-1}\mathcal{H}\mathcal{I}^{-1} = \mathcal{H}^{-1}. \tag{3.7}$$

These two alternative definitions are related by[20] $\mathcal{G} = \mathcal{I}^{-1}\mathcal{H}$, so that the compatibility condition in eq. (3.7) indeed corresponds to $\mathcal{G}^2 = \mathbb{1}$.

3.3 Dorfman and Courant brackets

Recall that the integrability of an (ordinary) almost complex structure is defined with respect to the Lie bracket, which acts on the space of vector fields. In order to carry the notion of integrability through to the realm of generalised geometry, we must introduce new brackets on such spaces of generalised vector fields.

Given sections $\mathbb{X} = X + \xi$ and $\mathbb{Y} = Y + \upsilon$ of the generalised tangent bundle, where $X, Y \in \Gamma(T)$ and $\xi, \upsilon \in \Gamma(T^*)$, we can define a generalised Lie derivative as follows.

---

[19]Taking the coset of a group by its maximal compact subgroup has an important physical implication. Consider two elements $X = X^a t_a$ and $Y = Y^a t_a$ of the Lie algebra $\mathfrak{g}$. There exists a natural pairing, the Cartan Killing metric, on $\mathfrak{g}$ with components $g_{ab} = \mathrm{tr}\; t_a t_b$, so that

$$\begin{aligned} g(X,Y) &= \mathrm{tr}(XY) \\ &= X^a Y^B \mathrm{tr}(t_a t_b) \\ &= X^a Y^b g_{ab}. \end{aligned}$$

For a non-compact G, this metric will be indefinite. However, reducing to the coset G/H, where H is the maximal compact subgroup of G, discards the directions in the Lie algebra associated with negative eigenvalues of g_{ab}, so that the metric induced on coset space by the Cartan Killing metric is positive definite. If G/H appears as the target space of a σ-model, the positive-definiteness of g_{ab} ensures that kinetic terms of the form $g_{ab} dx^a dx^b$ have the right sign.

[20]In index notation, this relation takes the form $\mathcal{H}_{KJ} = \mathcal{I}_{KI}\mathcal{G}^I{}_J$, where the positioning of the two indices on $\mathcal{G}$ follows from $\mathcal{G}$ being an automorphism of the generalised tangent bundle.

DEFINITION. The **Dorfman derivative** is the map

$$\begin{aligned} \mathbb{L}: \quad \Gamma(T\oplus T^*)\times\Gamma(T\oplus T^*) &\to \Gamma(T\oplus T^*) \\ \mathbb{X},\mathbb{Y} &\mapsto \mathbb{L}_{\mathbb{X}}\mathbb{Y} \equiv \mathcal{L}_X Y + \mathcal{L}_X \upsilon - \imath_Y d\xi, \end{aligned} \tag{3.8}$$

where $\mathcal{L}$ denotes the usual Lie derivative.

We can define the action of the Dorfman derivative onto a function f via an anchor map $a: T\oplus T^* \to T$. This in turn allows us to extend the action of the Dorfman derivative onto any tensor bundle.

The Dorfman derivative satisfies

$$\begin{aligned} \mathbb{L}_{\mathbb{X}}(\mathbb{L}_{\mathbb{Y}}\mathbb{Z}) &= \mathbb{L}_{\mathbb{L}_{\mathbb{X}}\mathbb{Y}}\mathbb{Z} + \mathbb{L}_{\mathbb{Y}}(\mathbb{L}_{\mathbb{X}}\mathbb{Z}) \\ \text{and} \quad \mathbb{L}_{\mathbb{X}}(f\mathbb{Y}) &= a(\mathbb{X})(f)\mathbb{Y} + f\mathbb{L}_{\mathbb{X}}\mathbb{Y}, \end{aligned} \tag{3.9}$$

which makes $(T\oplus T^*, \mathcal{I}, a, \mathbb{L})$ a Leibniz algebroid.

With this extension to general tensor bundles, the Dorfman derivative on the generalised metric $\mathcal{H}$ defined in section 3.2 can be written in a coordinate free fashion as[21]

$$(\mathbb{L}_{\mathbb{X}}\mathcal{H})(\mathbb{Y},\mathbb{Z}) = \mathbb{L}_{\mathbb{X}}(\mathcal{H}(\mathbb{Y},\mathbb{Z})) - \mathcal{H}(\mathbb{L}_{\mathbb{X}}\mathbb{Y},\mathbb{Z}) - \mathcal{H}(\mathbb{Y},\mathbb{L}_{\mathbb{X}}\mathbb{Z}). \tag{3.10}$$

We note that we may twist the Dorfman derivative by a closed 3-form H to give

$$\mathbb{L}^H_{\mathbb{X}}\mathbb{Y} \equiv \mathbb{L}_{\mathbb{X}}\mathbb{Y} + \imath_Y\imath_X H. \tag{3.11}$$

The significance of this and other twisted expressions in this section will become clear in section 3.9.

The Dorfman bracket is not antisymmetric; its symmetric part, however, is exact [25]:

$$\mathbb{L}_{\mathbb{X}}\mathbb{Y} + \mathbb{L}_{\mathbb{Y}}\mathbb{X} = 2d\mathcal{I}(\mathbb{X},\mathbb{Y}).$$

It will prove useful to define the antisymmetrisation of the Dorfman bracket.

[21] This is completely analogous to $(\mathcal{L}_X g)(Y,Z) = X(g(Y,Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z)$ for a metric g and vector fields X, Y, and Z.

Definition. The **Courant bracket** is given by

$$[\![\mathbb{X}, \mathbb{Y}]\!] \equiv [X, Y] + \mathcal{L}_X \upsilon - \mathcal{L}_Y \xi - \frac{1}{2} d(\imath_X \upsilon - \imath_Y \xi) \tag{3.12}$$

where [,] is the usual Lie bracket.

Note that the Courant bracket does not satisfy the Jacobi identity; its failure to do so is measured by the Jacobiator

$$\mathrm{Jac}(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = [\![[\![\mathbb{X}, \mathbb{Y}]\!], \mathbb{Z}]\!] + \text{cyclic}.$$

In particular, it can be shown (see for instance [17]) that

$$\mathrm{Jac}(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = dN(\mathbb{X}, \mathbb{Y}, \mathbb{Z}),$$

where the Nijenhuis tensor on generalised vector fields is [26]

$$N(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = \langle [\![\mathbb{X}, \mathbb{Y}]\!], \mathbb{Z} \rangle + \text{cyclic}.$$

As anticipated, the Courant bracket is related to the Dorfman derivative by

$$[\![\mathbb{X}, \mathbb{Y}]\!] = \frac{1}{2}(\mathbb{L}_{\mathbb{X}} \mathbb{Y} - \mathbb{L}_{\mathbb{Y}} \mathbb{X}).$$

The tuple $(T \oplus T^*, \mathcal{I}, a, [\![\ ,\ ]\!])$ consisting of a vector bundle endowed with a non-degenerate inner product, a skew-symmetric bracket, and a smooth bundle map, defines a Courant algebroid [4].

We can also introduce the Courant bracket as a *derived* bracket[22] via the operator expression [13]

$$[\![\mathbb{X}, \mathbb{Y}]\!] \cdot \equiv \frac{1}{2} \left([\{\mathbb{X}\cdot, d\}, \mathbb{Y}\cdot] - [\{\mathbb{Y}\cdot, d\}, \mathbb{X}\cdot] \right), \tag{3.13}$$

where $\mathbb{X}\cdot$ refers to the natural action of $\mathbb{X}$ on forms, which we will define in eq. (3.22).

[22] This is also in analogy with the case of the Lie bracket, which can be defined as a derived bracket via

$$[\{\imath_X, d\}, \imath_Y] = \imath_{[X,Y]}.$$

This formulation has the advantage that it generalises to multivectors, with the bracket on the right-hand side being the Schouten-Nijenhuis bracket [27].

We briefly pause to mention that these brackets result from the specialisation of more general structures. Formally (see for instance [27]), one can construct a derived bracket $[\ ,\ ]_d$ such that

$$[a,b]_d = [[a,d],b],$$

for $a,b \in \mathrm{End}(\Omega^\bullet(M))$ in the algebra of graded endomorphisms of the space $\Omega^\bullet(M)$ of differential forms, where $[\ ,\ ]$ is the graded commutator and d the usual de Rham differential. A derived bracket built in this way can be shown to be a Loday bracket, while its skew-symmetrisation, often labelled $[\ ,\ ]_d^-$ in the mathematical literature, is known as the Vinogradov bracket. Restricting $[\ ,\ ]_d$ to the direct sum of the spaces of vector fields and 1-forms recovers the explicit form of the Dorfman bracket in eq. (3.8), while restricting its skew-symmetrisation to the same space yields the Courant bracket. Here, we adopt the terminology whereby "derived" is used for both the genuine derived bracket, and its skew-symmetrisation [28].

For later use, we also introduce an H-twisted Courant bracket,

$$[\![\mathbb{X},\mathbb{Y}]\!]_H \equiv [\![\mathbb{X},\mathbb{Y}]\!] + \imath_X \imath_Y H, \tag{3.14}$$

where H is again a closed three-form, which we will later take to be the curvature of B, so that locally $H = dB$. Unsurprisingly, this twisted Courant bracket is a derived bracket [13]

$$[\![\mathbb{X},\mathbb{Y}]\!]_H \cdot \equiv \frac{1}{2}\left([\{\mathbb{X}\cdot, d_H\}, \mathbb{Y}\cdot] - [\{\mathbb{Y}\cdot, d_H\}, \mathbb{X}\cdot]\right), \tag{3.15}$$

with respect to the twisted differential[23]

$$d_H \equiv d - H\wedge, \tag{3.16}$$

which we note can be built out of the untwisted differential by conjugating with the exponentiated SO(d,d)-adjoint elements corresponding to B-shifts [15, 29],

$$d_H = e^B d e^{-B}.$$

Indeed, eq. (3.13) maps to eq. (3.15) under the twist $d \to d_H$.

Under a B-transformation,

$$[\![e^B\mathbb{X}, e^B\mathbb{Y}]\!]_{H-dB} = e^B[\![\mathbb{X},\mathbb{Y}]\!]_H, \tag{3.17}$$

so that, using terminology that will be explained in greater detail in section 3.9, $H^3(M)$ parametrises inequivalent twists.

One may wonder why discussions on generalised geometry often summon two different brackets, while in the context of ordinary differential geometry, a single bracket – the Lie bracket – suffices. To answer this, we should recall the two roles played by the Lie bracket in ordinary geometry. Firstly, the Lie bracket appears in the Lie algebra of diffeomorphisms: two vector fields (to which we associate two flows) commute to give a third vector field given by their Lie bracket. Secondly, the Lie derivative yields the infinitesimal transformation of a tensor under a diffeomorphism.

Upon generalising these concepts to generalised geometry, we are met with the need to define two different brackets. The Courant bracket appears in the gauge algebra of generalised diffeomorphisms, as we will see for instance in eq. (4.24). As

[23] Note that we require that H be closed under d for d_H to square to zero and therefore be a suitable differential; indeed, as an operator expression,

$$\begin{aligned} d_H^2 &= \frac{1}{2}\{d_H, d_H\} \\ &= \frac{1}{2}\{d,d\} + \{d, H\wedge\} + \frac{1}{2}\{H\wedge, H\wedge\} \\ &= dH\wedge, \end{aligned}$$

where we used $\{d,d\} = 0$ and $\{H\wedge, H\wedge\} = 0$.

expected, this bracket is antisymmetric. On the other hand, the Dorfman bracket takes up the second role of the usual Lie derivative; for instance, in eq. (4.25), we will see that it is the Dorfman derivative that appears in the definition of what we may call a "generalised isometry".

Returning to our original rationale for introducing these generalisations of the Lie bracket, we will see that the Courant bracket, as well as its twisted version, provide an appropriate notion of integrability for algebraic structures built on $T \oplus T^*$.

We now turn our attention to studying the symmetries of the Courant bracket. We define such a symmetry to be a bundle map F such that [17]

$$F([\![\mathbb{X}, \mathbb{Y}]\!]) = [\![F(\mathbb{X}), F(\mathbb{Y})]\!] \qquad \forall\, \mathbb{X}, \mathbb{Y} \in \Gamma(T \oplus T^*) \tag{3.18}$$

and such that the natural pairing is preserved. This bundle automorphism can be represented by the diagram

$$\begin{tikzcd} T \oplus T^* \arrow[r, "F"] \arrow[d, "\pi"'] & T \oplus T^* \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M. \end{tikzcd}$$

The diffeomorphisms of M, denoted $\mathrm{Diff}(M)$ and given by the bundle map

$$F = f_* \oplus f^*,$$

form part of the symmetry group of the Courant bracket. The action of $\mathrm{Diff}(M)$ is realised as that of matrix in eq. (3.2) on (sections of) the generalised tangent bundle. This is reminiscent of the symmetries of the Lie bracket[24].

[24]Indeed, these can also be represented by a diagram

$$\begin{tikzcd} T \arrow[r, "F"] \arrow[d, "\pi"'] & T \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M, \end{tikzcd}$$

with the condition $F([X, Y]) = [F(X), F(Y)]$ defining a symmetry of the Lie bracket, for vector fields X and Y. Here, the diffeomorphisms are given by $F = f_*$, and constitute the only symmetry of the Lie bracket [17].

However, there is now an additional symmetry: the B-transformation in eqs. (3.3) and (3.4), with B now a closed 2-form. Indeed, under such a transformation, the resulting change in the Courant bracket is

$$\begin{aligned}[\![X+\xi, Y+\upsilon]\!] \to e^B([\![X+\xi, Y+\upsilon]\!]) &= [\![X+\xi, Y+\upsilon]\!] + \imath_{[X,Y]}B \\ &= [\![X+\xi, Y+\upsilon]\!] + \\ &\quad + (d\imath_X\imath_Y + \imath_X d\imath_Y - \imath_Y d\imath_X - \imath_Y\imath_X d)B,\end{aligned}$$

where we used $\imath_{[X,Y]} = [\mathcal{L}_X, \imath_Y]$, followed by Cartan's magic formula, $\mathcal{L}_X = d\imath_X + \imath_X d$. The above result should be compared with the Courant bracket of the B-transformed generalised vectors,

$$\begin{aligned}[\![e^B(X+\xi), e^B(Y+\upsilon)]\!] &= [\![X+\xi+\imath_X B,\ Y+\upsilon+\imath_Y B]\!] \\ &= [\![X+\xi, Y+\upsilon]\!] + \mathcal{L}_X\imath_Y B - \mathcal{L}_Y\imath_X B - \frac{1}{2}d(\imath_X\imath_Y B - \imath_Y\imath_X B) \\ &= [\![X+\xi, Y+\upsilon]\!] + (d\imath_X\imath_Y + \imath_X d\imath_Y - \imath_Y d\imath_X)B,\end{aligned}$$

where again we used Cartan's magic formula, as well as the anticommutativity of the interior product. We thus see that

$$e^B([\![X+\xi, Y+\upsilon]\!]) = [\![e^B(X+\xi), e^B(Y+\upsilon)]\!] + \imath_X\imath_Y dB,$$

where $dB = 0$ since B is closed – implying that the B-transform in eq. (3.4) is a symmetry of the Courant bracket. We see, then, that the automorphism group of the Courant algebroid [30] is precisely the geometric subgroup in eq. (3.5). Locally, then,

$$\mathrm{GDiff} \sim \mathrm{GL}(d;\mathbb{R}) \ltimes \Omega^2_{\mathrm{ex}},$$

where Ω^2_{ex} is the space of exact two-forms. Note that these are precisely the transformations generated by the Dorfman bracket (as evident from eq. (3.8)); this is equivalent to the property (eq. (3.9)) that the Dorfman derivative acts as a derivation on itself.

3.4 Generalised (almost) complex structures

We proceed on our journey of generalising the algebraic structures commonly encountered in complex differential geometry with the following definition.

DEFINITION. A **generalised almost complex structure** $\mathcal{J}$ is an endomorphism

$$\mathcal{J} : T \oplus T^* \to T \oplus T^*$$

which satisfies $\mathcal{J}^2 = -\mathrm{id}$ and preserves the natural pairing metric, i.e.

$$\mathcal{I}(\mathcal{J}\mathbb{X}, \mathcal{J}\mathbb{Y}) = \mathcal{I}(\mathbb{X}, \mathbb{Y}) \qquad \forall\ \mathbb{X}, \mathbb{Y} \in \Gamma(T \oplus T^*). \tag{3.19}$$

This last condition can be equivalently written as $\mathcal{J}^T\mathcal{I} = -\mathcal{I}\mathcal{J}$, and so amounts to the statement that the canonical metric $\mathcal{I}$ is Hermitian with respect to $\mathcal{J}$ [12].

In analogy with eq. (2.3), we introduce projectors

$$\Pi_{\pm} = \frac{1}{2}(\mathbb{1} \pm i\mathcal{J}),$$

which resolve the (complexified) generalised tangent bundle into

$$(T \oplus T^*) \otimes \mathbb{C} = \mathbb{L} \oplus \bar{\mathbb{L}},$$

where $\mathbb{L} = \Pi_+\mathbb{L}$ and $\bar{\mathbb{L}} = \Pi_-\bar{\mathbb{L}}$ are the $\pm i$-eigenspaces of $\mathcal{J}$.

The condition that the canonical metric be Hermitian, eq. (3.19), then implies that the subbundles $\mathbb{L}, \bar{\mathbb{L}}$ are (maximally) isotropic [17], since

$$\begin{aligned}
\langle \Pi_\pm \mathbb{X}, \Pi_\pm \mathbb{Y} \rangle &= \mathbb{X}^T \Pi_\pm^T \mathcal{I} \Pi_\pm \mathbb{Y} \\
&= \frac{1}{4}\mathbb{X}^T(\mathcal{I} \pm i\mathcal{J}^T\mathcal{I} \pm i\mathcal{I}\mathcal{J} - \mathcal{J}^T\mathcal{I}\mathcal{J})\mathbb{Y} \\
&= 0.
\end{aligned}$$

It can be shown that, in general, a generalised almost complex structure takes the form

$$\mathcal{J} = \begin{pmatrix} A & P \\ L & -A^T \end{pmatrix},$$

where A is a $(1,1)$-tensor, P a bivector, and L and 2-form [17].

In analogy with that of an ordinary complex structure, the integrability of a generalised almost complex structure coincides with the involutility of its eigenbundles $\mathbb{L}, \bar{\mathbb{L}}$ under the Courant bracket [31]:

$$\Pi_{\mp}[\![\Pi_{\pm}\mathbb{X}, \Pi_{\pm}\mathbb{Y}]\!] = 0 \qquad \forall\ \mathbb{X}, \mathbb{Y} \in \Gamma(T \oplus T^*). \tag{3.20}$$

Again, this condition can be recast as the vanishing of the Nijenhuis tensor.

The specification of a generalised almost complex structure concurs with a reduction of the structure group from $\mathrm{O}(d,d)$ down to $\mathrm{U}(d/2, d/2) = \mathrm{O}(d,d) \cap \mathrm{GL}(d, \mathbb{C})$ [4].

The formalism above elegantly encompasses complex and symplectic structures; specifically, these correspond to the cases

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix} \qquad \text{and} \qquad \mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}, \tag{3.21}$$

respectively, where J is a complex structure and ω a symplectic one. The $\mathbb{L}$ eigenbundles in each case are $T^{0,1} \oplus T^{*1,0}$ and $\{X - i\imath_X\omega, V \in \Gamma(T) \otimes \mathbb{C}\}$, with associated integrability conditions $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$ (i.e. J is itself integrable) and $d\omega = 0$ [4]. We thus see that generalised (almost) complex structures attractively interpolate between complex structures and symplectic ones[25].

Finally, we note that we can twist the notion of integrability into that of H-integrability[26]; that is, a generalised almost complex structure $\mathcal{J}$ whose $\mathbb{L}$ eigenbundle satisfies the involutility condition in eq. (3.20) with the usual Courant bracket $[\![\ ,\ ]\!]$ replaced by its H-twisted version, $[\![\ ,\ ]\!]_H$, defined in eq. (3.14). For instance, $\mathcal{J}_J$ in eq. (3.21) is H-integrable if H is a $(2,1) \oplus (1,2)$-form (and if J is integrable too) [12].

[25] In fact, it can be shown that, on a manifold with a complex structure, we can always take local coordinates that are part complex, and part symplectic – in other words, the neighbourhood around any (regular) point corresponds to the product of open sets in the complex space and in the symplectic space. This result is known as the generalised Darboux theorem [12].

[26] In this case, the Darboux theorem (see footnote 25) states that regular neigbourhoods correspond to the B-transform (with B not closed) of the product of complex and symplectic open sets [4].

3.5 Generalised Kähler geometries

We now present an important example of a generalised geometry. We begin by recalling two results presented in the previous sections: equipping a generalised tangent bundle with a generalised metric $\mathcal{G}$ reduces the structure group from $\mathrm{O}(d,d)$ to $\mathrm{O}(d) \times \mathrm{O}(d)$, while equipping it with a generalised complex structure $\mathcal{J}$ breaks $\mathrm{O}(d,d)$ down to $\mathrm{U}(d/2, d/2)$.

Consider now a generalised complex structure $\mathcal{J}_1$. Supplying the generalised tangent bundle with a generalised metric $\mathcal{G}$, on top of and compatible[27] with $\mathcal{J}_1$, provokes a further collapse of the structure group from $\mathrm{U}(d/2, d/2)$ to its maximal compact subgroup, $\mathrm{U}(d/2) \times \mathrm{U}(d/2)$, to which $\mathrm{U}(d/2, d/2)$ is homotopic [4]. Together, $\mathcal{G}$ and $\mathcal{J}_1$ are said to define a generalised Hermitian structure [24].

In fact, given $\mathcal{G}$ and $\mathcal{J}_1$, we automatically have a second generalised almost complex structure $\mathcal{J}_2 = \mathcal{G}\mathcal{J}_1$. The requirement that $\mathcal{J}_2^2 = -\mathbb{1}$ is evidently met by $\mathcal{G}^2 = \mathbb{1}$, $\mathcal{J}_1^2 = -\mathbb{1}$, and the compatibility condition $\mathcal{G}\mathcal{J}_1 = \mathcal{J}_1\mathcal{G}$. Therefore, we see that only two of $(\mathcal{G}, \mathcal{J}_1, \mathcal{J}_2)$ are independent. In particular, the requirement that $\mathcal{G}^2 = 1$ translates into the requirement that $\mathcal{J}_1$ and $\mathcal{J}_2$ commute.

Imposing that both $\mathcal{J}_1$ and $\mathcal{J}_2$ be integrable yields a generalisation of the Kähler condition.

Definition. A **generalised Kähler structure** is a pair of commuting generalised complex structures $(\mathcal{J}_1, \mathcal{J}_2)$, such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is positive-definite [4].

A *twisted* generalised Kähler structure is one whose generalised complex structures are H-integrable.

3.6 Polyforms and generalised spinors

We take the lift from an $\mathrm{O}(d,d)$ structure to $\mathrm{Spin}(d,d)$. Note that – unlike the lift from $\mathrm{O}(d)$ to $\mathrm{Spin}(d)$, which we recall had topological conditions attached to it,

[27] By compatibility we mean that the generalised complex structure and the generalised metric commute with each other. In other words, we take the C_+ eigenbundle of $\mathcal{G}$ to be stable under $\mathcal{J}_1$.

namely the manifold being a spin manifold – this new lift carries no conditions with it[28]. We denote the resulting spin bundle as S; we will now argue that this can be associated with $\Lambda^\bullet T^*$, the bundle of polyforms ϕ – formal sums of differential forms of different degree. This relies on the existence of a natural action of the generalised vector $\mathbb{X} = X + \xi \in \Gamma(T \oplus T^*)$ on polyforms, namely

$$\mathbb{X} \cdot \phi = \imath_X \phi + \xi \wedge \phi. \tag{3.22}$$

In particular, if we take a generalised coordinate vector field

$$\tilde{\mathbb{X}} = \frac{\partial}{\partial x^n} + dx^m,$$

made up of a coordinate vector field and a coordinate covector field, for some fixed indices n and m, its action on a polyform is

$$\tilde{\mathbb{X}} \cdot \phi = \left(\Gamma_n + \Gamma^m\right) \phi,$$

where[29,30]

$$\Gamma^m = dx^m \wedge \qquad \text{and} \qquad \Gamma_n = \imath_n$$

satisfy

$$\{\Gamma^m, \Gamma^n\} = 0, \qquad \{\Gamma^m, \Gamma_n\} = \delta^m_n, \qquad \text{and} \qquad \{\Gamma_m, \Gamma_n\} = 0,$$

and so provide a representation of the Clifford(d,d) algebra in terms of forms[31] [23].

This can be generalised [6] by defining a map

$$\begin{array}{rccc} \Gamma_{\mathbb{X}}: & \Lambda^\bullet T^* & \to & \Lambda^\bullet T^* \\ & \phi & \mapsto & \Gamma_{\mathbb{X}}(\phi) \equiv \mathbb{X} \cdot \phi \end{array}$$

[28] We are perhaps being too hasty here. It is indeed true that $\mathrm{O}(d,d)$ structures can always be lifted to $\mathrm{Spin}(d,d)$ ones in the case of $T \oplus T^*$ – even if the underlying manifold is not a spin manifold. However, for general bundles, this is only possible if the second Stiefel-Whitney classes of the subbundles $C_\pm$ of section 3.2 are identical [4, 6]

[29] Note that Γ^m and Γ_n are not related to each other by the raising or lowering of an index via some metric.

[30] We use the notation $\imath_n \equiv \imath_{\partial/\partial x^n}$.

[31] The Clifford(d) algebra for the usual spinors on T, $\{\gamma^m, \gamma^n\} = 2g^{mn}\mathbb{1}$, finds a representation in $\gamma^m = dx^m \wedge + g^{mn}\imath_n$.

and then noting that

$$\begin{aligned}\{\Gamma_{\mathbb{X}}, \Gamma_{\mathbb{Y}}\}\ (\phi) &= \imath_X \imath_Y \phi + \xi \wedge \imath_Y \phi + \imath_X (\upsilon \wedge \phi) + \xi \wedge \upsilon \wedge \phi + \\ &\qquad + \imath_Y \imath_X \phi + \upsilon \wedge \imath_X \phi + \imath_Y (\xi \wedge \phi) + \upsilon \wedge \xi \wedge \phi \\ &= (\imath_X \upsilon + \imath_Y \xi)\phi \\ &= (\xi(Y) + \upsilon(X))\phi \\ &= 2\langle \mathbb{X}, \mathbb{Y} \rangle \phi \\ &= 2\mathcal{I}(\mathbb{X}, \mathbb{Y})\phi,\end{aligned}$$

a Clifford algebra. In the above, we used that interior products anticommute $\{\imath_X, \imath_Y\} = 0$, as well as the graded Leibniz rule $\imath_X(\alpha \wedge \beta) = \imath_X \alpha \wedge \beta + (-1)^k \alpha \wedge \imath_X \beta$ for a k-form α, and the notation $\imath_X \omega = X^\mu \omega_\mu = \omega(X)$ for a one-form ω. This shows that we can revisit the space $\Lambda^\bullet T^*$ of polyforms and regard it as an irreducible module for the Clifford algebra bundle $\mathrm{Cl}(T \oplus T^*)$; generalised vectors can be seen as the gamma matrices associated to the natural pairing metric $\mathcal{I}$ [12,24]. This parallelism between the Clifford algebra and the action of generalised vectors onto polyforms allows us to translate various geometrical structures into the language of spinors.

We proceed by embedding the Majorana and Weyl conditions on spinors into the architecture of differential forms. The Majorana condition amounts to the limitation to polyforms that are real. The Weyl condition, on the other hand, separates the exterior algebra into even and odd forms. Polyforms made up of either only even or only odd forms correspond respectively to spinors of positive and negative chirality [12]. We label the bundles of spinors with positive or negative chirality as $S^\pm$, and those of even or odd polyforms as $\Lambda^\pm T^*$.

The above discussion might be suggestive of a relation $S^\pm = \Lambda^\pm T^*$. In fact, a more careful analysis shows that the action of $\mathrm{GL}(n;\mathbb{R}) \subset \mathrm{Spin}(n,n)$ on $\Lambda^\bullet T^*$ is [4]

$$\phi \mapsto \sqrt{|\mathrm{det} N|}\ N \cdot \phi,$$

where $N \cdot \phi$ is the usual action of $\mathrm{GL}(n;\mathbb{R})$ on $\Lambda^\bullet T^*$, for $N \in \mathrm{GL}(n;\mathbb{R})$. Therefore, we find

$$S = \Lambda^\bullet T^* \otimes \sqrt{\det\, T}. \tag{3.23}$$

Making a choice of trivialisation[32] [32], we have a non-canonical spinor-polyform isomorphism [6]

$$S^{\pm} \simeq \Lambda^{\pm} T^{*}.$$

The Clifford map offers a concrete realisation of the isomorphism between the positive- and negative-chirality spinor bundles and the space of even and odd polyforms [33]. To see how, we write a Clifford(d,d) spinor[33] $\Phi^{\pm}$ as a tensor product [15] of Clifford(d) spinors $\eta^{1,2}$,

$$\Phi^{\pm} \sim \eta^{1}_{+} \otimes \eta^{2\dagger}_{\pm}, \tag{3.24}$$

then use the Fierz identities to write

$$\eta^{1}_{+} \otimes \eta^{2\dagger}_{\pm} = \frac{1}{8} \sum_{k} \frac{1}{k!} \eta^{2\dagger}_{\pm} \gamma_{i_1 \ldots i_k} \eta^{1}_{+} \gamma^{i_k \ldots i_1},$$

and finally employ the Clifford map [13]

$$\alpha \equiv \sum_{k} \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} dx^{i_i} \wedge \ldots \wedge dx^{i_k} \quad \longleftrightarrow \quad \not{\alpha} \equiv \sum_{k} \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \gamma^{i_i \ldots i_k}_{\alpha\beta}, \tag{3.25}$$

to identify Clifford(d,d) spinors with polyforms. In fact, the slash notation is often dropped to avoid cluttering.

From now on, motivated by the parallelism described above, we will use the terms “polyforms” and “(generalised) spinors” interchangeably.

The usual bilinear form on spinors finds its manifestation on the Clifford module in the $\Lambda^{d} T^{*}$-valued Mukai pairing between polyforms ϕ_1 and ϕ_2,

$$(\phi_1, \phi_2) \equiv (\sigma(\phi_1) \wedge \phi_2)|_{\text{top}} \tag{3.26}$$

where

$$\sigma(\phi) = (-1)^{[n/2]} \phi, \tag{3.27}$$

[ ] takes the integer value, and $|_{\text{top}}$ projects the top form part. We note that the

[32] As we will see, this is related to the dilaton.

[33] The reason for the notation employed here will become apparent in the next chapter.

above bilinear form on $\Lambda^\bullet T^*$ is compatible with the Clifford action,

$$(\mathbb{X}\cdot\phi_1, \mathbb{X}\cdot\phi_2) = \langle\mathbb{X},\mathbb{X}\rangle(\phi_1,\phi_2)$$

for all generalised vectors $\mathbb{X}$, so that it is invariant under B-transformations [24],

$$(e^B\phi_1, e^B\phi_2) = (\phi_1,\phi_2).$$

We now introduce another facet of the relationship between spinors and geometrical structures. First, let us give the following definitions.

Definition. The **null space** of a complex spinor $\phi \in (\Lambda^\bullet T^*M)\otimes\mathbb{C}$ is the subbundle[34]

$$L_\phi = \{\mathbb{X}\in\Gamma(T\oplus T^*)\otimes\mathbb{C} \mid \mathbb{X}\cdot\phi = 0\} \tag{3.28}$$

of the complexified generalised tangent bundle consisting of the annihilators $\mathbb{X}$ of ϕ.

We can refine further the above definition.

Definition. A **pure spinor**[35] Φ is a spinor whose annihilator space L_Φ is maximal:

$$\mathrm{rank}(L_\Phi) = \frac{1}{2}\mathrm{rank}(T\oplus T^*). \tag{3.29}$$

The algebraic association

$$L_\Phi = \mathbb{L}_\mathcal{J} \tag{3.30}$$

between the null space of a pure spinor and the $+i$-eigenbundle $\mathbb{L}_\mathcal{J}$ of $\mathcal{J}$ allows

[34] Note that this subbundle is isotropic, i.e. $\mathcal{I}(\mathbb{X},\mathbb{Y}) = 0$ for any two annihilators $\mathbb{X},\mathbb{Y}$. This can be seen via

$$2\mathcal{I}(\mathbb{X},\mathbb{Y})\phi = (\mathbb{X}\mathbb{Y}+\mathbb{Y}\mathbb{X})\cdot\phi = 0,$$

for $\mathbb{X},\mathbb{Y}\in L_\phi$, using the Clifford algebra [4].

[35] An alternative (and equivalent) definition is that a spinor η is pure if half of the gamma matrices annihilate it, i.e. if $\eta^T\gamma_{i_1\dots i_m}\eta = 0$ for $m < d/2$ [34]. In six dimensions, every Weyl spinor is pure.

us to identify generalised almost complex structures with lines[36] of pure spinors, and viceversa. This correspondence is particularly useful, as generalised complex structures and pure spinors arise most clearly from quite different perspectives – those of the worldsheet and of the spacetime, respectively [22].

The B-transformation $B \cdot \phi = B \wedge \phi$ of a polyform $\phi \in \Gamma(\Lambda^\bullet T^*)$ exponentiates to

$$\phi \to \phi_D \equiv e^B \phi = \left(1 + B + \frac{1}{2} B \wedge B + \dots\right) \wedge \phi,$$

where, in this context, ϕ is sometimes referred to as a "naked" spinor, while the rotated ϕ_D is referred to as a "dressed" spinor [4, 15].

3.7 Generalised Calabi-Yau geometries

We now discuss a generalised geometry which will play a fundamental role in the following discussions on string compactifications.

Definition. A **generalised Calabi-Yau structure** is a pure spinor Φ that is closed, i.e. $d\Phi = 0$, and that satisfies[37] $(\Phi, \bar{\Phi}) \neq 0$ everywhere.

Note that there is no universal consensus on the definition of a generalised Calabi-Yau structure. To distinguish it from alternative definitions, one of which will be presented later, the structure defined above is sometimes referred to as a *weak* generalised Calabi-Yau [17], or as a generalised Calabi-Yau *à la Hitchin* [12].

The fact that the canonical bundle K admits a non-vanishing, global closed section $\Phi \in \Gamma(K)$ implies that it is holomorphically trivial [4, 24]. In terms of the generalised almost complex structure $\mathcal{J}$ associated with Φ, the closure of Φ is equivalent to the integrability of $\mathcal{J}$. In fact,

$$\mathcal{J} \text{ integrable} \qquad \Leftrightarrow \qquad d\Phi = \mathbb{W} \cdot \Phi, \tag{3.31}$$

[36] Complex pure spinors differing only by an overall scale factor share the same annihilator space, and so correspond to the same generalised almost complex structure. More precisely, then, the equivalence is between a generalised almost complex structure and a complex line subbundle $K \subset (\Lambda^\bullet T^*) \otimes \mathbb{C}$, referred to as the *canonical* line subbundle [4, 24].

[37] $\bar{\Phi}$ is the complex conjugate of Φ.

for some generalised vector $\mathbb{W}$. The overall factor of Φ does not enter the above condition, and so in the presence of a globally defined pure spinor, we take $\mathbb{W} = 0$ [12]. The condition above is the generalisation of eq. (2.8) for ordinary complex structures.

To see how the correspondence in eq. (3.31) arises, we note that, for $\mathbb{X}, \mathbb{Y} \in L_\Phi = \mathbb{L}_\mathcal{J}$, the action of the Courant bracket in its derived form (eq. (3.13)) becomes

$$\begin{aligned} [\![\mathbb{X}, \mathbb{Y}]\!] \cdot \Phi &= (\mathbb{X}\mathbb{Y} - \mathbb{Y}\mathbb{X}) \cdot d\Phi \\ &= (\mathbb{X}\mathbb{Y} - \mathbb{Y}\mathbb{X})\, \mathbb{W} \cdot \Phi \\ &= 0. \end{aligned}$$

The last equality is trivial if $\mathbb{W} = 0$, but it also holds if $\mathbb{W} \neq 0$, since acting with one creator $\mathbb{W}$ on the Clifford vacuum Φ and then acting with two annihilators $\mathbb{X}$ and $\mathbb{Y}$ always yields zero. From the above, then, it follows that $[\![\mathbb{X}, \mathbb{Y}]\!] \cdot \Phi = 0$ and so $[\![\mathbb{X}, \mathbb{Y}]\!] \in L_\Phi = L_\mathcal{J}$: the eigenbundle of $\mathcal{J}$ is Courant involutive, and so $\mathcal{J}$ is integrable [13, 35]. This proves the correspondence in eq. (3.31).

We may twist the above statements, and in particular recast the H-twisted integrability of a generalised almost complex structure into the condition that its associated pure spinor satisfies $d_H\Phi = \mathbb{W} \cdot \Phi$. An H-twisted generalised Calabi-Yau structure is then given by a pure spinor that satisfies $(\Phi, \bar{\Phi}) \neq 0$, and that is closed under d_H [15].

We have seen that, on a d-dimensional manifold, a generalised almost complex structure $\mathcal{J}$ entails a reduction of the structure group to $\mathrm{U}(d/2, d/2)$ and is associated to the canonical line bundle $K \subset (\Lambda^\bullet T^*) \otimes \mathbb{C}$ (defined in footnote 36) whose Clifford annihilator is the $\mathbb{L}_\mathcal{J}$ eigenbundle of $\mathcal{J}$. The existence of this subbundle generates the following decomposition of polyforms:

$$\Lambda^\bullet T^* \otimes \mathbb{C} = \bigoplus_{k=-d/2}^{d/2} U_k,$$

where we identify the top degree component $U_{d/2}$ with the canonical line bundle K, and we define the other subbundles as $U_k = \Lambda^{d/2-k}\bar{\mathbb{L}}_\mathcal{J} \cdot K$, so that U_k is the ik-eigenbundle of (the Lie algebra action of) $\mathcal{J}$ [24]. A *local* section of K is a pure spinor. As we already mentioned, if K is holomorphically trivial, it admits a non-vanishing *global* section Φ such that

$$(\Phi, \bar{\Phi}) \neq 0.$$

This fixes the ambiguity in the overall factor of the pure spinor corresponding to $\mathcal{J}$; in this case, then, we observe a further reduction of the structure group from $\mathrm{U}(n/2, n/2)$ to $\mathrm{SU}(n/2, n/2)$ [12]. Thus, generalised Calabi-Yau structures are $\mathrm{SU}(n/2, n/2)$ structures. We can see this by noting that

$$\frac{\Phi}{\sqrt{|(\Phi, \overline{\Phi})|}}$$

is a non-vanishing section of the complexified tangent bundle

$$S^{\pm} \otimes \mathbb{C} \simeq \Lambda^{\pm} T^* \otimes \sqrt{\Lambda^d T} \otimes \mathbb{C}.$$

Since $L_\Phi \subset (T \oplus T^*) \otimes \mathbb{C}$ is maximally isotropic, we may equivalently write

$$S^{\pm} \otimes \mathbb{C} \simeq \Lambda^{\pm} L_\Phi^* \otimes \sqrt{\Lambda^d L_\Phi} \otimes \mathbb{C},$$

and since $\Phi \in \Gamma(\sqrt{\Lambda^d L_\Phi} \otimes \mathbb{C})$, then the global trivialisation of $\Lambda^d L_\Phi^*$ obtained by squaring Φ gives a volume form [1]. This finally implies that the structure group is reduced further from $\mathrm{U}(n/2, n/2)$ to $\mathrm{SU}(n/2, n/2)$.

There is an alternative definition of a Calabi-Yau structure.

Definition. A **generalised Calabi-Yau *metric* structure** is a pair of pure spinors $\Phi_1, \Phi_2 \in \Gamma(\Lambda^\bullet T^* \otimes \mathbb{C})$, such that

$$d\Phi_1 = d\Phi_2 = 0 \qquad \text{and} \qquad (\Phi_1, \bar{\Phi}_1) = \alpha(\Phi_2, \bar{\Phi}_2) \neq 0,$$

for non-zero constant α, and such that their associated generalised complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ form a generalised Kähler structure [36].

From our earlier discussion, it follows that an equivalent definition of a generalised Calabi-Yau metric structure is that of a generalised Kähler structure whose two generalised complex structures each have a holomorphically trivial canonical bundle [4]. These structures are sometimes referred to as generalised Calabi-Yau structures *à la Gualtieri* [12].

Finally, we may once more twist this definition into that of a twisted generalised Calabi-Yau metric structure by twisting the differential, $d \to d_H$.

3.8 Torsion classes for SU(3)×SU(3) structures

In section 2.5, we reviewed the torsion classes in the case of SU(3) structures. Here, we will extend this analysis to SU(3)×SU(3) structures, which will play a crucial role in the discussion of flux compactifications.

It turns out that the same information contained in eqs. (2.22) and (2.23) can be displayed in the form of the covariant derivative of an invariant spinor η defining an SU(3) structure [37],

$$\nabla_m \eta = i q_m \gamma_7 \eta + i q_{mn} \gamma^n \eta,$$

where $\gamma_7 = -i\varepsilon_{mnpqrs}\gamma^{mnpqrs}/6!$. In other words, (q_m, q_{mn}) can be mapped to ($\mathcal{W}_1$, $\mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5$), and viceversa. Similarly, for SU(3)×SU(3) structures,

$$\nabla_m \eta^i_+ = i q^i_m \eta^i_+ + i q^i_{mn} \gamma^n \eta^i_-,$$

from which we can obtain [38]

$$\begin{aligned} d\Phi_+ &= W^{10}_m \gamma^m \Phi_+ + W^{01}_m \Phi_+ \gamma^m + W^{30} \bar{\Phi}_- + W^{21}_{mn} \gamma^m \bar{\Phi}_- \gamma^n \\ &\quad + W^{12}_{mn} \gamma^m \Phi_- \gamma^n + W^{03} \Phi_-, \end{aligned} \tag{3.32a}$$

$$\text{and} \quad \begin{aligned} d\Phi_- &= W^{13}_m \gamma^m \Phi_- + W^{02}_m \Phi_- \gamma^m + W^{33} \bar{\Phi}_+ + W^{22}_{mn} \gamma^m \bar{\Phi}_+ \gamma^n \\ &\quad + W^{11}_{mn} \gamma^m \Phi_+ \gamma^n + W^{00} \Phi_+. \end{aligned} \tag{3.32b}$$

Positive chirality Spin(6,6) spinors can be decomposed under Spin(6)×Spin(6) as

$$\mathbf{32}^+ = (\mathbf{4}, \mathbf{4}) \oplus (\overline{\mathbf{4}}, \overline{\mathbf{4}}),$$

and under SU(3) $\subset$ Spin(6) we further have $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$. This gives a total of 8 SU(3)×SU(3) representations. We can choose the following basis, arranged into a diamond:

$$\begin{array}{ccccccc}
 & & & \Phi_+ & & & \\
 & & \gamma^m\Phi_+ & & \Phi_+\gamma^n & & \\
 & \gamma^m\bar{\Phi}_- & & \gamma^m\Phi_+\gamma^n & & \Phi_-\gamma^m & \\
\bar{\Phi}_- & & \gamma^m\bar{\Phi}_-\gamma^n & & \gamma^m\Phi_-\gamma^n & & \Phi_- \\
 & \bar{\Phi}_-\gamma^m & & \gamma^m\bar{\Phi}_+\gamma^n & & \gamma^m\Phi_- & \\
 & & \bar{\Phi}_+\gamma^m & & \gamma^m\bar{\Phi}_+ & & \\
 & & & \bar{\Phi}_+ & & &
\end{array} \tag{3.33}$$

where Φ_+ corresponds to the $(\mathbf{1}, \bar{\mathbf{1}})$ module of SU(3)×SU(3), $\gamma^m\Phi_+$ to $(\bar{\mathbf{3}}, \bar{\mathbf{1}})$, $\Phi_+\gamma^m$ to $(\mathbf{1}, \mathbf{3})$, and so on[38] [33]. This retrospectively justifies the notation for the components W^{ij} in eq. (3.32) – the superscripts label the position within the diamond above.

3.9 Twisting with a gerbe

Finally, we address an issue which we mentioned in multiple instances – that of twisting. Physically, this discussion will turn out to be relevant in the case of vacua admitting Neveu-Schwarz fluxes – or, in the present language, in the case of non-closed B – and in fact can be generalised to account for Ramond-Ramond fluxes as well.

If H is non-trivial, then dressed spinors[39] are no longer global sections of (the spinor bundle over) $T \oplus T^*$ [22]: B is only defined locally[40], and we must allow for gauge transformations

$$B_{(\alpha)} - B_{(\beta)} = d\lambda_{(\alpha\beta)}$$

on twofold patch intersections $U_\alpha \cap U_\beta$, where $\lambda_{(\alpha\beta)} = -\lambda_{(\beta\alpha)}$ is a one form. This is accompanied by the consistency condition that $\lambda_{(\alpha\beta)} + \lambda_{(\beta\gamma)} + \lambda_{(\gamma\alpha)}$ be closed, and therefore exact, on the threefold overlap $U_\alpha \cap U_\beta \cap U_\gamma$. If the flux is quantised, i.e.

[38] By $\bar{\mathbf{1}}$ we mean the singlet appearing in the decomposition of $\bar{\mathbf{4}}$, as opposed to that of $\mathbf{4}$.

[39] More precisely, spinors transformed by a non-closed B.

[40] To emphasize the local nature of B, the notation "$H = dB_{(\alpha)}$ on every patch U_α" is sometimes used in place of $H = dB$ [12].

if $H \in H^3(M, \mathbb{Z})$ represents an integral cohomology class, then we have that

$$\lambda_{(\alpha\beta)} + \lambda_{(\beta\gamma)} + \lambda_{(\gamma\alpha)} = g^{-1}_{(\alpha\beta\gamma)} dg_{(\alpha\beta\gamma)},$$

where the U(1)-valued transition functions $g_{(\alpha\beta\gamma)} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$ observe the cocycle condition

$$g_{(\beta\gamma\delta)} g^{-1}_{(\alpha\gamma\delta)} g_{(\alpha\beta\delta)} g^{-1}_{(\alpha\beta\gamma)} = 1$$

on the overlap $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$, as well as $g_{(\alpha\beta\gamma)} = g^{-1}_{(\beta\alpha\gamma)}$. We then say that B is a connection on a gerbe, while H is the curvature of the gerbe connection [6, 7, 39].

We can now define a new bundle E over the manifold M by considering $T \oplus T^*$ and introducing the identification

$$\mathbb{X}_{(\alpha)} = e^{-d\lambda_{(\alpha\beta)}} \mathbb{X}_{(\beta)}$$

on the overlap $U_\alpha \cap U_\beta$, or, in components,

$$X_{(\alpha)} = X_{(\beta)} \quad \text{and} \quad \xi_{(\alpha)} = \xi_{(\beta)} + \imath_{X_{(\beta)}} d\lambda_{(\alpha\beta)}.$$

In other words, the generalised vectors $\mathbb{X}$ are in fact local sections of the twisted bundle E given by the particular extension

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0.$$

E therefore encapsulates both the topological properties of T and the connective structure of a gerbe given by building a fibration of T^* over T with two-form shifts, on top of the usual patching by diffeomorphisms. We may then write the overall patching as

$$X_{(\alpha)} = A_{(\alpha\beta)} \cdot X_{(\beta)} \quad \text{and} \quad \xi_{(\alpha)} = A^{-T}_{(\alpha\beta)} \xi_{(\beta)} + \imath_{A_{(\alpha\beta)} \cdot X_{(\beta)}} d\lambda_{(\alpha\beta)}$$

on the overlap $U_\alpha \cap U_\beta$, where $A_{(\alpha\beta)} \in \mathrm{GL}(d)$ enacts diffeomorphisms.

There is a Leibniz algebroid isomorphism between $(T \oplus T^*, \mathbb{L}^H)$ and $(E, \mathbb{L})$ [11]. Therefore, the untwisted and twisted pictures are equivalent: in the former, we restrict to naked pure spinors (or spinors dressed at most with a closed B), and H enters explicitly in the various constructions over $T \oplus T^*$, for example via twisted integrability, the H-twisted Courant bracket $[\![\ ,\ ]\!]_H$, and the twisted differential d_H; in the latter, we consider dressed spinors on E, and work with the usual (untwisted) notions of integrability, Courant bracket $[\![\ ,\ ]\!]$, and differential d [15]. Due to their equivalence, we will use the two pictures interchangeably.

Indeed, by performing an appropriate B-transform in each patch, we may "eliminate" the H field from eq. (3.17), and proceed with the newly untwisted Courant bracket. The gerbe structure is now encoded into the generalised tangent bundle itself.

In this new twisted tangent bundle, the fibre over a point p in the base manifold is still $T_p \oplus T_p^*$. However, the twisting implies that the transition functions now also include B-transformations. We have therefore enlarged the structure group of the bundle to the geometric subgroup $\mathrm{GL}(d;\mathbb{R}) \ltimes \Omega^2_{\mathrm{cl}}$ in eq. (3.5) by introducing the action of exact 2-forms. We note, however, that this is still only a part of the larger $\mathrm{O}(d,d)$ group [6].

Chapter 4

Generalised geometry in supergravity compactifications

4.1 Introduction to compactifications

Superstring theories exist critically only in 10 dimensions. Contending with the observation that our spacetime geometry is 4-dimensional, it is natural to wonder where these extra dimensions may be. A popular resolution is to compactify the 6 extra dimensions – vaguely speaking, curling them up on themselves – so that they are effectively unobservable at low energies. Geometrically, the 10-dimensional space is fibred over a 4-dimensional space, which we take to be our own spacetime. Remarkably, the geometry of these extra dimensions has a profound effect on our four-dimensional physics. The intimate interaction between our everyday world and the hidden dimensions is an active area of research. In this chapter, we review how the language of generalised geometry provides a natural description of these ideas.

4.1.1 Kaluza-Klein compactifications in field theory

Prior to delving into string theory, it is instructive to discuss Kaluza-Klein compactifications in the context of ordinary field theory. In particular, we consider a massless scalar field $\phi(y, x)$ propagating in a d-dimensional direct-product spacetime

$$M_{d-1} \times S^1,$$

taking y^μ, with $\mu \in \{0, \dots, d-2\}$, and x to be the coordinates on M_{d-1} and S^1, respectively, so that we have the identification $x \sim x + 2\pi R$, where R is the radius of the transverse space S^1. Using this periodicity in x, we can Fourier expand the d-dimensional field as

$$\phi(y, x) = \sum_{k=-\infty}^{\infty} \varphi_k(y) e^{ikx/R}.$$

By doing so, we generate an infinite tower of modes φ_k indexed by k (which will turn out to be related to the momentum in the compact direction, as we will see just below). Using this decomposition, we can expand the d-dimensional Klein-Gordon field equation for ϕ,

$$\Box\phi(y, x) = 0,$$

where $\Box = \partial_y^2 + \partial_x^2$, into[41]

$$\left(\partial_y^2 - \frac{k^2}{R^2}\right) \varphi_k(y) = 0,$$

revealing that $\{\varphi_k\}$ is an infinite tower of massive modes (for $k \neq 0$), together with a massless zero mode, φ_0. Furthermore, the masses

$$m^2 = \frac{k^2}{R^2}$$

are quantised, since $k \in \mathbb{Z}$. The momentum along the compact direction x is then[42] $p^x = k/R$.

For energies $E \ll 1/R$, the massive modes “decouple” from the theory: in the limit $R \to 0$, an infinite amount of energy is required to excite any one of them, rendering the (massless) ground state the only observable state of the theory. Thus, at low energies, we see the emergence of $(d-1)$-dimensional physics from a d-dimensional theory [40].

We now turn to the effects of compactification on objects with non-trivial Lorentz quantum numbers. Compared to the scalar case, there is now an additional step,

[41]Recall Fourier series are expansions in orthogonal bases, so from the wave equation of ϕ we find a wave equation for ϕ_k for each value of k.

[42]Note that we use x as a superscript (and later as a subscript too) to label the component of the d-dimensional momentum vector along the compactified direction – it is not a power.

namely the decomposition of representations of SO(d) into ones of the lower-dimensional SO($d-1$). Each object spawned by this decomposition can then be Kaluza-Klein reduced to generate an infinite tower of modes standing on top of a massless ground state.

For example, a metric g_{MN}, where $M, N \in \{0, \ldots, d-1\}$, descends into the lower-dimensional theory to give a metric $g_{\mu\nu}$, a vector field $g_{\mu x} = g_{x\mu}$, and a scalar field g_{xx}, all living in M_{d-1}. For each of these, we can Fourier expand the compact coordinate to give a Kaluza-Klein tower of excitations. Similarly, a vector field $A_M dX^M$, where $X^M = (y^\mu, x)$, produces a lower-dimensional vector field $A_\mu dy^\mu$, and a scalar field $A_x dx$.

The metric $g_{\mu\nu}$ clearly inherits a $(d-1)$-dimensional diffeomorphism invariance within the non-compact space from the d-dimensional one of the full theory. However, the full theory is invariant under reparametrisations of the compact coordinate as well,

$$x = x + \lambda(y^\mu). \tag{4.1}$$

This symmetry manifests itself in the lower-dimensional theory in the form of invariance under the gauge transformation

$$g_{\mu x} \to g_{\mu x} - \partial_\mu \lambda, \tag{4.2}$$

which can be seen by imposing the invariance of the line element [40]

$$ds^2 = g_{MN} dX^M dX^N = g_{\mu\nu} dy^\mu dy^\nu + g_{xx}(dx + g_{\mu x} dy^\mu)^2$$

under eq. (4.1). We will refer to the U(1) symmetry in eq. (4.2) as the Kaluza-Klein gauge symmetry. As this gauge symmetry is the realisation of spacetime translations on the compactification manifold, its charge is the internal momentum k/R.

We thus reach a striking conclusion: the gauge symmetry in the lower-dimensional theory arises as a vestige of the diffeomorphism invariance in higher dimensions; this is how the Kaluza-Klein mechanism spawns gauge bosons in the non-compact directions out of the zero modes of the d-dimensional graviton on the compact manifold.

A similar story sees an antisymmetric tensor B_{MN} get dissected into[43] the an-

[43]In compactifications on n-dimensional tori, there are also $n(n-1)/2$ scalars [22].

tisymmetric tensor $B_{\mu\nu}$ and the vector field $B_{x\mu}$, with the invariance of the higher-dimensional theory under

$$B_{MN} \to B_{MN} + \delta B_{MN}, \qquad \text{where} \qquad \delta B_{MN} = \partial_M \zeta_N - \partial_N \zeta_M,$$

carrying over to the lower-dimensional theory in the form of the invariance under antisymmetric transformations $\delta B_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu$ with parameter ζ_μ, as well as a new U(1) gauge symmetry with field $B_{x\mu}$ and parameter ζ_x.

However, at least within the bounds of field theory, fields cannot carry charge under this second gauge transformation. This is due to the fact that there are no states which are charged under the 2-form field B_{MN} in the higher-dimensional theory [40]. In section 4.1.2, we will see that winding states in string theory are in fact charged under this gauge field.

By assembling the reduced fields into an effective action, one can see that the vacuum expectation value of the scalar g_{xx}, which is related to R, has no potential. The compactification radius, therefore, is not fixed by the equations of motion – any background is consistent, regardless of the value of R. Such fields parameterising flat directions of the potential are therefore massless (c.f. Goldstone bosons), and are referred to as moduli [23, 40]. Unlike Goldstone bosons, however, degenerate states are not mapped into each other by symmetries of the theory. In other words, the vacuum expectation values of these moduli distinguish physically inequivalent vacua [41]. The presence of moduli in a theory is problematic for (at least) two reasons. First of all, moduli hinder the predictive power of a theory, as couplings would be expected to depend on the unfixed vacuum expectation values [22]. Secondly, such massless fields would lead to a long-range interaction of strength comparable to that of gravity; this would imply deviations from the equivalence principle which have not been observed to date [23]. Therefore, "stabilising" the moduli by generating potentials for them is an important objective for any realistic theory. As we will mention, an attractive feature of flux compactifications is that they lead to a stabilisation of (at least some of) the moduli.

We now discuss the compactification of d-dimensional pure gravity (a "type-0" theory) on $M_{d-n} \times T^n$. We take a d-dimensional metric g_{MN} which, upon compactification, yields a metric $g_{\mu\nu}$, n vector fields $g_{\mu m}$, and $n(n+1)/2$ scalars[44] g_{mn}, where $\mu, \nu \in \{0, \dots, d-n-1\}$ and $m, n \in \{1, \dots, n\}$. Restricting to the massless sector, i.e. the massless modes of the Kaluza-Klein towers, the theory has a $\mathrm{GL}(n; \mathbb{R})$ symmetry group. Considering the whole Kaluza-Klein towers, this is broken down to the discrete $\mathrm{GL}(n; \mathbb{Z})$. As we will now see, this has a very natural origin.

By an identification of the type $x \sim x + 2\pi q R$ in each of the compact directions, where $q \in \mathbb{Z}$, we can view $T^n = \mathbb{R}^n / \mathbb{Z}^n$. The natural action of $\mathrm{GL}(n; \mathbb{R})$ on $\mathbb{R}^n$, namely $v^i \to M^i{}_j v^j$ for $v \in \mathbb{R}^n$ and $M \in \mathrm{GL}(n; \mathbb{R})$, carries over to T^n provided that $M \in \mathrm{GL}(n; \mathbb{Z}) \subset \mathrm{GL}(n; \mathbb{R})$, so that the action of M preserves the equivalence classes defining the quotient group. Just like $\mathrm{GL}(n; \mathbb{R})$ is the group of diffeomorphisms of $\mathbb{R}^n$, so $\mathrm{GL}(n; \mathbb{Z})$ is the diffeomorphism group of T^n. These are often referred to as "large" diffeomorphisms due to the fact that, unlike other diffeomorphisms, they are not generated by the exponentiation of infinitesimal transformations. We thus see that $\mathrm{GL}(n; \mathbb{Z})$, the symmetry group of the Kaluza-Klein reduced theory, has a natural interpretation in terms of the diffeomorphisms on the tori.

4.1.2 Toroidal compactifications in string theory and T-duality

We now investigate compactifications in the context of string theory. In comparison to the field theory case explored in the previous section, we will find a much richer structure of symmetries stemming from the extended nature of strings.

As always, we take the worldsheet fields for bosonic string theory, which we arrange as $X^M = (Y^\mu, X)$, to be coordinates on spacetime, the target space, but we now assume that the X direction is compactified. For the non-compact directions, we have the usual identification in terms of worldsheet coordinates (σ, τ),

$$Y^\mu(\sigma + 2\pi, \tau) = Y^\mu(\sigma, \tau), \tag{4.3}$$

where $\sigma \in [0, 2\pi)$. This is also true for the compact direction (in fact, only for non-winding modes, as we will see shortly). As in field theory, the momentum $p^x = k/R$ – here, the momentum of the centre of mass of the string state – along

[44] These parametrise the coset space $\mathrm{GL}(n; \mathbb{R})/\mathrm{SO}(n)$.

the X direction is quantised, $k \in \mathbb{Z}$, so that

$$e^{i\delta X k/R} = 1 \qquad \text{for} \qquad \delta X = 2\pi R.$$

What is new compared to the field theory case is that strings, being extended objects, can *wrap around* the compact direction, so that along this direction, the identification in eq. (4.3) actually takes the form

$$X(\sigma + \pi, \tau) = X(\sigma, \tau) + 2\pi R w,$$

where $w \in \mathbb{Z}$ is the winding number – strings can "feel" topology to a deeper extent than point particles can.

Again, compactification equips the lower-dimensional theory with two gauge symmetries: the Kaluza-Klein gauge symmetry, which realises the higher-dimensional reparametrisation invariance of the compact direction, and the gauge symmetry originating from the antisymmetric 2-form field. The two gauge bosons $g_{\mu x}$ and $B_{\mu x}$ are, respectively,

$$(\alpha^{\mu}_{-1}\tilde{\alpha}^{x}_{-1} \pm \alpha^{x}_{-1}\tilde{\alpha}^{\mu}_{-1})|0, p\rangle,$$

where α and $\tilde{\alpha}$ are the usual creation operators in bosonic string theory [41].

By introducing a complex parametrisation

$$z = \tau + i\sigma, \qquad \bar{z} = \tau - i\sigma, \tag{4.4}$$

with the corresponding derivatives

$$\partial = \frac{1}{2}(\partial_\tau - i\partial_\sigma), \qquad \bar{\partial} = \frac{1}{2}(\partial_\tau + i\partial_\sigma), \tag{4.5}$$

the vertex operators for the two gauge fields can be written as

$$V_{\pm}(p) \sim \int d^2z\ \zeta_\mu(\partial Y^\mu \bar{\partial} X \pm \partial X \bar{\partial} Y^\mu) e^{ip_\mu Y^\mu},$$

where we omitted normalisation and string coupling. We see that the vertex operator of the second gauge boson is a total derivative,

$$\partial Y^\mu \bar{\partial} X - \partial X \bar{\partial} Y^\mu = \bar{\partial}(X \partial Y^\mu) - \partial(X \bar{\partial} Y^\mu).$$

Coupling a state to the zero-momentum vertex operators above measures its charge under the corresponding gauge symmetries. The second vertex operator being a total derivative, it is clear that field theory states cannot be charged under this symmetry. In string theory, however, the total derivative above does not always integrate to zero, since X is multi-valued for strings within the non-zero winding sector [41].

Given the vertex operator of a tachyon with compact momentum k and winding w,

$$V_{\text{tac}}(p) \sim \int d^2 z \; e^{ip_\mu Y^\mu} e^{ip_L \cdot X(z) + ip_R \cdot \overline{X}(\bar{z})},$$

where

$$p_L = \frac{k}{R} + \frac{wR}{\alpha'} \quad \text{and} \quad p_R = \frac{k}{R} - \frac{wR}{\alpha'},$$

the 3-point amplitude for two tachyons (with charges (k, w) and $(-k, -w)$) coupling to a gauge boson is [42]

$$\langle V_\pm(p_1) V_{\text{tac}}(p_2) V_{\text{tac}}(p_3) \rangle \sim (2\pi)^{d-1} \delta^{d-1}\big(\sum_i p_i\big) \zeta_\mu p_{23}^\mu (p_L \pm p_R),$$

where we defined $p_{23}^\mu = p_2^\mu - p_3^\mu$. The gauge coupling of a tachyon under each of the two gauge symmetries is given by taking the momentum p_1 of the gauge boson to zero [41]. We find that the charge under the Kaluza-Klein $g_{\mu x}$ gauge field is $p_L + p_R \sim k/R$. This matches with our earlier field-theoretical analysis. We also see that the charge under the $B_{\mu x}$ gauge field is $p_L - p_R \sim wR/\alpha'$: winding states are charged under the gauge symmetry descending from the d-dimensional antisymmetric tensor B_{MB}. This is in contrast to field theory states, which we recall could only carry the first of the $\mathrm{U}(1) \times \mathrm{U}(1)$ charges. We mention in passing that, at the self-dual radius $R = \sqrt{\alpha'}$, stringy effects imply that the actual gauge symmetry is enhanced to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ [41].

The mass-shell condition consists in the string spectrum being

$$m^2 = \frac{k^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2).$$

We note that this is completely symmetric under

$$R \leftrightarrow \frac{\alpha'}{R}, \qquad k \leftrightarrow w; \tag{4.6}$$

indeed, as $R \to \infty$, winding modes become infinitely massive and the momentum spectrum becomes continuous, while the opposite behaviour occurs as $R \to 0$, implying a complete duality between string theory at radii R and α'/R. Specifically, the physics at radius R is equivalent to that at radius α'/R under the interchange of winding and momenta modes. It can be shown (for example, at the level of the partition function) that this symmetry extends to higher genera, and therefore is a symmetry of the full, interacting theory[45].

This target space symmetry is referred to as T-duality and it is a purely stringy phenomenon [41].

4.1.3 Narain compactification

The (p_L, p_R) lattice can be shown to be even with respect to the inner product with (d, d) signature,

$$(p_L, p_R) \cdot (p'_L, p'_R) = \alpha'(p^i_L p_{L,i} - p^i_R p_{R,i}) = 2(k^i w'_i + w^i k'_i) \in \mathbb{Z},$$

and is also self-dual[46].

Given any even, self-dual Lorentzian lattice, all other such lattices can be generated by acting with $\mathrm{O}(d, d; \mathbb{R})$ transformations on it. However, acting separately with two different $\mathrm{O}(d; \mathbb{R}) \times \mathrm{O}(d; \mathbb{R}) \subset \mathrm{O}(d, d; \mathbb{R})$ transformations on the same lattice produces two physically equivalent backgrounds. Furthermore, as we saw earlier, T-duality implies that transformations belonging in the discrete subgroup $\mathrm{O}(d, d; \mathbb{Z}) \subset \mathrm{O}(d, d; \mathbb{R})$ consist of permutations that leave the lattice unchanged

[45] In fact, this is only true provided the vacuum expectation value of the dilaton is shifted by an appropriate amount. We will briefly return to this issue in section 4.4.

[46] This is related to the modular invariance of the 1-loop partition function [40].

overall. Therefore, we reach the conclusion that the moduli space, the space of inequivalent string theories is [40]

$$\frac{\mathrm{O}(d,d;\mathbb{R})}{\mathrm{O}(d;\mathbb{R})\times\mathrm{O}(d;\mathbb{R})\times\mathrm{O}(d,d;\mathbb{Z})}. \tag{4.7}$$

The above coset space, modulo the T-duality identification, will turn out to be parametrised by a Riemannian metric and a 2-form, meaning that the NSNS background can be seen as the set of rotation parameters in the above space (again, ignoring the $\mathrm{O}(d,d;\mathbb{Z})$ factor).

4.2 Generalised geometry and the NSNS sector

We will now see that the Riemannian metric g and the Neveu-Schwarz-Neveu-Schwarz 2-form $B \in \Lambda^2 T^*$ naturally merge into a generalised metric, as defined in section 3.2. This is the first true instance in which we apply the language of generalised geometry to describe string theory.

First of all, we note that

$$\mathcal{G}_0 = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

constitutes a positive-definite metric on $T \oplus T^*$. In fact, the above metric can be obtained from an ordinary Kähler structure, whose description we recall contains a Riemannian structure g, a complex structure J, and a symplectic structure ω. We gave the corresponding generalised complex structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$ in eq. (3.21). We find that

$$-\mathcal{J}_\omega \mathcal{J}_J = \begin{pmatrix} 0 & J\omega^{-1} \\ -\omega J & 0 \end{pmatrix} = \mathcal{G}_0,$$

where $g = -\omega J$ is the usual metric. Given that $\mathcal{J}_J$ and $\mathcal{J}_\omega$ commute, and that $\mathcal{G}_0$ is positive definite, we can conclude that we are in the presence of a generalised Kähler structure, as defined in section 3.5.

In fact, the B-transform $\{e^B \mathcal{J}_J e^{-B}, e^B \mathcal{J}_\omega e^{-B}\}$ of a generalised Kähler structure $\{\mathcal{J}_J, \mathcal{J}_\omega\}$ is again a generalised Kähler structure, for B closed [4]. We can thus parametrise a generic generalised metric as

$$\begin{aligned}\mathcal{G} &= e^{B}\mathcal{G}_0 e^{-B} \\ &= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \\ &= \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}.\end{aligned} \tag{4.8}$$

As for the generalised metric $\mathcal{H}$, also defined in section 3.2, we find it is given by the $2d \times 2d$ matrix

$$\begin{aligned}\mathcal{H} &= \mathcal{I}^{-1}\mathcal{G} \\ &= \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix},\end{aligned} \tag{4.9}$$

with respect to which the norm of a generalised vector $\mathbb{X} = X + \xi$ is

$$\mathcal{H}(\mathbb{X}, \mathbb{X}) = g(X, X) + g^{*}(\xi + \imath_X B, \xi + \imath_X B),$$

g^{*} being the metric on T^{*} obtained by inverting g.

We have seen that the specification of a generalised metric $\mathcal{H}$ is equivalent to that of a Riemannian metric g and a 2-form B. These two fields parametrise the coset space (familiar from eq. (4.7))

$$\frac{\mathrm{O}(d,d)}{\mathrm{O}(d) \times \mathrm{O}(d)}, \tag{4.10}$$

which is equivalent to the statement (which we recall from section 3.2) that the generalised metric $\mathcal{H}$ defines a reduction of the structure group on $T \oplus T^{*}$ from $\mathrm{O}(d,d)$ to $\mathrm{O}(d) \times \mathrm{O}(d)$. Indeed, at any point on the manifold, the space of such reductions is precisely the coset space above. In other – perhaps somewhat more suggestive – words, we have an association

$$\text{Neveu-Schwarz-Neveu-Schwarz sector} \quad \Leftrightarrow \quad \mathrm{O}(d) \times \mathrm{O}(d) \text{ structure.}$$

In the discussion above, we ignored the dilaton. We will see it is naturally embedded into the formalism of string theory via the normalisation of the pure spinors defining a generalised Calabi-Yau metric structure.

We end this section by mentioning the relation between generalised Kähler structures and the bihermitian geometries of Gates-Hull-Roček. The latter are defined by data $(g, H, J_\pm)$, where H is a closed 3-form, the metric g is Hermitian ($J_\pm^T g J_\pm = g$) with respect to the complex structures $J_\pm$, and

$$\nabla^\pm J_\pm = 0,$$

where the Bismut connections $\nabla^\pm$ are given by

$$\nabla^\pm = \nabla \pm \frac{1}{2} g^{-1} H,$$

∇ being the Levi-Civita connection [17]. In [43] it was found that, given a supersymmetric non-linear σ-model with $\mathcal{N} = (1,1)$ worldsheet supersymmetry, the presence of an additional left-moving supersymmetry and of an additional right-moving one requires the target space to have a bihermitian geometry. This is then a necessary condition for the existence of supersymmetry on spacetime given a non-zero Neveu-Schwarz-Neveu-Schwarz flux. Remarkably, in [4], it was shown that the Gates-Hull-Roček geometry is equivalent to the generalised Kähler geometry. In fact, the generalised complex structures $\mathcal{J}_{1,2}$ defining the generalised Kähler geometry are given in terms of the data $(g, H, J_\pm)$ by

$$\mathcal{J}_{1,2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_+ \pm J_- & -\left(\omega_+^{-1} \mp \omega_-^{-1}\right) \\ \omega_+ \mp \omega_- & -\left(J_+^t \pm J_-^t\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \tag{4.11}$$

where $H = dB$ and the 2-forms $\omega_\pm = g J_\pm$. Equation (4.11) is often referred to as the Gualtieri map [36].

In the next sections, we will encounter more instances of supersymmetry conditions finding natural interpretations in the language of generalised geometry.

4.3 Compactifications on T^n and T-duality

We now return to provide a somewhat more thorough treatment of the symmetry that is T-duality. We will see the emergence of the generalised metric $\mathcal{H}$ of eq. (4.9) and the coset space of eq. (4.10). We will articulate our discussion by first introducing T-duality in the context of higher-dimensional toroidal compactifications. We will then briefly discuss the appearance of T-duality in curved target space geometries in section 4.4. In section 4.5 we will translate T-duality in the language of generalised geometry. The objective is to provide a brief presentation of how covariance with respect to the symmetries of string theory is built into the formalism of generalised geometry.

Consider a string theory compactified on $M_{d-n} \times T^n$. It is useful to assemble winding numbers w^m and masses k_n, where $m, n \in \{1, \ldots, n\}$, into an $\mathrm{O}(n, n; \mathbb{Z})$ vector

$$Z^I = (w^m, k_m), \qquad \text{where } I \in \{1, \ldots, 2n\}.$$

It can be shown that the zero-mode Hamiltonian may be written as

$$H = \frac{1}{2}(p_L^2 + p_R^2) = \frac{1}{2}(Z^I)^T \mathcal{H}_{IJ} Z^J. \tag{4.12}$$

Anticipating the presence of symmetries that mix g and B, it will be useful to define a background matrix

$$E = g + B \in \frac{\mathrm{O}(n, n; \mathbb{R})}{\mathrm{O}(n; \mathbb{R}) \times \mathrm{O}(n; \mathbb{R})}$$

and embed it into $\mathrm{O}(n, n; \mathbb{R})$ by defining the action of a general element[47]

$$\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}(n, n; \mathbb{R})$$

onto an $n \times n$ matrix M by the fractional linear transformation

$$\mathcal{O} \cdot M \equiv (aM + b)(cM + d)^{-1}. \tag{4.13}$$

[47] Here, a, b, c, and d are $n \times n$ matrices satisfying $a^T c = -c^T a$, $b^T d = -d^T b$, and $a^T d = 1 - c^T b$.

The transformation above correctly encodes that of $\mathcal{H}$ under $\mathrm{O}(n,n)$, namely

$$\mathcal{H} \to \mathcal{O}^T \mathcal{H} \mathcal{O},$$

for $\mathcal{O} \in \mathrm{O}(n,n)$.

We seek the symmetries of the spectrum in eq. (4.12). Consider a transformation under which

$$Z \to AZ, \tag{4.14}$$

where A is a $2n \times 2n$ matrix. In fact, for the quantisation of masses and winding numbers to be preserved, we need A to have integer entries. For this transformation to leave the spectrum invariant, we also require that $\mathcal{H}$ transform as

$$\mathcal{H} \to A^{-T} \mathcal{H} A^{-1},$$

where $A^{-T} = (A^{-1})^T$. Finally, we wish that the level-matching condition,

$$N - \tilde{N} = w^m k_m = \frac{1}{2} Z^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Z$$

be preserved under such a symmetry. Substituting for eq. (4.14), the requirement for the above condition to be left intact becomes that

$$A^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

implying that the symmetry group is $\mathrm{O}(n,n;\mathbb{Z})$. These symmetries arrange themselves into three families, as follows [44]:

- the "large" gauge transformations given by discrete shifts[48]

  $$B \to B + \Psi,$$

  where $\Psi_{ij} = -\Psi_{ji} \in \mathbb{Z}$: Ψ is an $n \times n$ antisymmetric matrix with integer

[48] This transformation shifts the action by $2\pi \int \Psi$, where Ψ is pulled back to the worldsheet. Being Ψ integer-valued, this shift does not alter the path integral, and therefore corresponds to a symmetry of the theory. We thus see that B plays the role of a theta parameter, providing nothing more than a topological contribution to the action [44].

entries. In terms of the embedding of the background matrix E, these transformations take the form

$$\mathcal{O} = \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix} \in \mathrm{O}(n, n; \mathbb{Z}).$$

Indeed, using the prescription in eq. (4.13), we see that this choice of $\mathcal{O}$ gives

$$E \to \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix} \cdot E = E + \Psi,$$

which amounts to a shift in B by Ψ, since Ψ is antisymmetric and B is the antisymmetric part of E.

- discrete changes of basis,

  $$E \to \Psi E \Psi^T, \qquad \text{where } \Psi \in \mathrm{GL}(n; \mathbb{Z}).$$

  Embedding $\mathrm{GL}(n;\mathbb{Z})$ into $\mathrm{O}(n,n;\mathbb{Z})$ (see eq. (3.2)), this transformation is given by

  $$\mathcal{O} = \begin{pmatrix} \Psi & 0 \\ 0 & (\Psi^T)^{-1} \end{pmatrix}.$$

  Indeed, with this $\mathcal{O}$, we see that

  $$E \to \begin{pmatrix} \Psi & 0 \\ 0 & (\Psi^T)^{-1} \end{pmatrix} \cdot E = \Psi E ((\Psi^T)^{-1})^{-1} = \Psi E \Psi^T;$$

- factorised dualities, with

  $$\mathcal{O} = \begin{pmatrix} 1 - e_i & e_i \\ e_i & 1 - e_i \end{pmatrix}, \tag{4.15}$$

  where $(e_i)_{mn} = \delta_{im}\delta_{in}$. These provide a generalisation of T-duality to various axes.

In conclusion, $\mathrm{O}(n,n;\mathbb{Z})$ is the T-duality group – the symmetry group of a bosonic string theory compactified on T^n.

4.4 T-duality in curved backgrounds

We now consider dualities on curved target space geometries. For simplicity, we will restrict our analysis to compact Abelian symmetries. A treatment of more sophisticated symmetries is available, for instance, in [26, 44, 45].

Consider a σ-model on a d-dimensional manifold with target space metric g_{ij}, 2-form B_{ij}, and dilaton $\phi(x)$. We take the action of the σ-model to be

$$S = \frac{1}{2\pi}\int d^2z(g_{\mu\nu}(x) + B_{\mu\nu}(x))\partial x^\mu \overline{\partial} x^\nu - \frac{1}{8\pi}\int d^2z\phi(x)R^{(2)} \tag{4.16}$$

where we introduced complex coordinates (and derivatives) on the worldsheet, as in eqs. (4.4) and (4.5).

We assume the existence of a U(1) isometry – that is, a Killing vector field X:

$$\mathcal{L}_X g_{\mu\nu} = \nabla_\nu k_\mu + \nabla_\mu k_\nu = 0. \tag{4.17}$$

For this to be a symmetry of the action S, we also require

$$\mathcal{L}_X H = 0, \tag{4.18}$$

where H is the field strength of the B field, i.e. $H = dB$ locally. Note that one does not need a condition as strong as $\mathcal{L}_X B = 0$ – in fact, eq. (4.18) locally allows for gauge transformations

$$B \to B' = B + d\xi',$$

so that in a general gauge we can write eq. (4.18) as

$$\mathcal{L}_X B - d\xi = 0, \tag{4.19}$$

where

$$\xi = -\mathcal{L}_X \xi' = -\imath_X d\xi' + df \tag{4.20}$$

for smooth functions f. In fact, for simplicity, we will now work in the gauge $\mathcal{L}_X B = 0$. Finally, for the dilaton term in S to be left unchanged too, we also require

$$X^\mu \partial_\mu \phi = 0.$$

We now take adapted coordinates[49] – that is, we split the coordinates as $x^\mu = \{\theta, \tilde{x}^i\}$, in such a way that the isometry corresponds to translations $\theta \to \theta + \epsilon$, relegating the $\tilde{x}^i$ to the role of spectator fields (on which the background fields may still depend). The action of eq. (4.16) reads, in this new coordinate system,

$$\begin{aligned} S\left[\theta, \tilde{x}^i\right] = \frac{1}{2\pi} \int d^2 z \big(& g_{00}\left(\tilde{x}\right) \partial\theta\bar{\partial}\theta + \left(g_{0i}\left(\tilde{x}\right) + B_{0i}\left(\tilde{x}\right)\right) \partial\theta\bar{\partial}\tilde{x}^i \\ & + \left(g_{i0}\left(\tilde{x}\right) + B_{i0}\left(\tilde{x}\right)\right) \partial\tilde{x}^i\bar{\partial}\theta + \left(g_{ij}\left(\tilde{x}\right) + B_{ij}\left(\tilde{x}\right)\right) \partial\tilde{x}^i\bar{\partial}\tilde{x}^j \big) \\ & - \frac{1}{8\pi} \int d^2 z \phi\left(\tilde{x}\right) R^{(2)}. \end{aligned}$$

We now gauge the isometry, making ϵ a function of the worldsheet coordinates, via minimal coupling. This provokes the introduction of gauge fields A and $\overline{A}$, with variations

$$\delta_\epsilon A = -\partial\epsilon \quad \text{and} \quad \delta_\epsilon \overline{A} = -\overline{\partial}\epsilon.$$

This gauging yields

$$\begin{aligned} S\left[\tilde{x}^i, A, \chi\right] = \frac{1}{2\pi} \int d^2 z \big(& g_{00} A\bar{A} + \left(g_{0i} + B_{0i}\right) A\bar{\partial}\tilde{x}^i + \left(g_{i0} + B_{i0}\right) \partial\tilde{x}^i \bar{A} \\ & + \left(g_{ij} + B_{ij}\right) \partial\tilde{x}^i\bar{\partial}\tilde{x}^j \big) - \frac{1}{8\pi} \int d^2 z (\phi(\tilde{x}) R^{(2)} + \chi F). \end{aligned}$$

A few explanations are in order. Firstly, the above action is given in a gauge where $\theta = 0$. The Lagrange multiplier χ sets, on-shell, the Abelian field strength

$$F = \partial\overline{A} - \overline{\partial}A,$$

to vanish, so that A and $\overline{A}$ are constrained[50] to be (locally) pure gauge on-shell, i.e. $A = \partial\theta$ and $\overline{A} = \overline{\partial}\theta$. Thus, we recover the original action[51] in eq. (4.16). Integrating by parts the Lagrangian multiplier, and eliminating the gauge fields – which by now

[49] These coordinates may not exist globally. If this is the case, one may dualise the theory patch by patch, and then assemble the dual background into a global one.

[50] We are implicitly restricting ourselves to worldsheets with trivial topology here.

[51] In the case of compact symmetries, the periodicity of θ is recovered by demanding that the holonomies $h = \oint A$ along homology cycles be integers n. For the canonical homology cycles of the torus, these correspond to the winding modes of χ.

are auxiliary fields and have algebraic equations of motion – we find

$$\hat{S} = \frac{1}{2\pi}\int d^2z \left(\hat{g}_{\mu\nu}(\tilde{x}) + \hat{B}_{\mu\nu}(\tilde{x})\,\partial y^\mu \bar{\partial} y^\nu\right) - \frac{1}{8\pi}\int d^2z \hat{\phi}(\tilde{x})\, R^{(2)},$$

where we introduced new coordinates $y^\mu = \{\chi, \tilde{x}^i\}$. Crucially, $\hat{S}$ is the action of a new theory, dual to that of the original σ-model. The relationship between the two dual models is,

$$\hat{g}_{00} = 1/g_{00}, \qquad \hat{g}_{0i} = B_{0i}/g_{00}, \qquad \hat{g}_{ij} = g_{ij} - (g_{i0}g_{0j} + B_{i0}B_{0j})/g_{00}, \tag{4.21a}$$

$$\hat{B}_{0i} = g_{0i}/g_{00}, \qquad \hat{B}_{ij} = B_{ij} - (g_{i0}B_{0j} + B_{i0}g_{0j})/g_{00}, \quad \text{and} \quad \hat{\phi} = \phi - \frac{1}{2}\log g_{00}, \tag{4.21b}$$

where the transformation of the dilaton appears as a factor in the measure in the integration of the gauge fields. The exact shift of the dilaton is chosen by demanding the dual theory be conformally invariant.

Equation (4.21) implies that the geometries of the dual theories are vastly different. In particular, it implies that the duality we described acts on the background as a factorised duality along θ, as defined in eq. (4.15).

The approach described above is due to Buscher [46, 47]. We mention in passing that this procedure can be generalised to non-Abelian symmetries and symmetries along fermionic directions. We can also relax the condition $\mathcal{L}_X B = 0$ to $\mathcal{L}_X H = 0$. This leads to a simple generalisation of the Buscher rules in eq. (4.21) (see, for instance, [45]).

4.5 T-duality and generalised geometry

Consider a bosonic background of string theory which admits an isometry generated by a Killing vector field X. As in the previous section, in order to perform a T-duality via the Buscher procedure, we must have that

$$\mathcal{L}_X g = 0, \tag{4.22a}$$

$$\text{and} \quad \mathcal{L}_X B - d\xi = 0. \tag{4.22b}$$

Looking at eqs. (4.22a) and (4.22b), we see that the action of the symmetries of the NS sector – diffeomorphisms and B-gauge transformations – is defined by the specification of a vector X and a covector ξ, which of course we can combine into a generalised vector $\mathbb{X} = X + \xi \in \Gamma(T \oplus T^*)$.

We thus see that the gauge parameters of the NS sector are sections of the generalised tangent bundle, $T \oplus T^*$ – an instance of how generalised geometry offers a covariantisation of the symmetries of string compactifications [26].

We can also show that the Courant bracket emerges rather naturally in this context. To see this, we denote the gauge transformation (see eq. (4.22b)) of the B-field parametrised by the generalised vector $\mathbb{X} = X + \xi$ as

$$\delta_{\mathbb{X}} B = \mathcal{L}_X B + d\xi. \tag{4.23}$$

Its closure (or rather, its closure defect) with another gauge transformation parametrised by $\mathbb{Y} = Y + \upsilon$ is then

$$[\delta_{\mathbb{X}}, \delta_{\mathbb{Y}}] B = \mathcal{L}_{[X,Y]} B + d(\mathcal{L}_X \upsilon - \mathcal{L}_Y \xi),$$

where we used $\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}$, and $\mathcal{L} d\alpha = d\mathcal{L}\alpha$ for any form α. We also dropped from the right-hand side a term containing $d\xi$ and $d\upsilon$ – we can do so by gauging via eq. (4.20) with $\xi' = \upsilon' = 0$. By comparing the right-hand side above with eq. (4.23), we write

$$\begin{aligned} [\delta_{\mathbb{X}}, \delta_{\mathbb{Y}}] &= \delta_{[X,Y] + \mathcal{L}_X \upsilon - \mathcal{L}_Y \xi} \\ &= \delta_{[X,Y] + \mathcal{L}_X \upsilon - \mathcal{L}_Y \xi - \frac{1}{2} d(\imath_X \upsilon - \imath_Y \xi)}, \end{aligned}$$

where we added an exact term which leaves the gauge transformation unaltered, since d is nilpotent. We can now see that the right-hand side is the gauge transformation of the B-field parametrised by the generalised vector given by the Courant bracket between $\mathbb{X}$ and $\mathbb{Y}$:

$$[\delta_{\mathbb{X}}, \delta_{\mathbb{Y}}] = \delta_{[\![\mathbb{X}, \mathbb{Y}]\!]}. \tag{4.24}$$

In other words, two gauge transformations of the B-field commute to give a gauge transformation along the Courant bracket between the original gauge parameters [26].

Using eq. (3.10), we find that the Dorfman derivative of the generalised metric $\mathcal{H}$ with respect to this generalised vector is [48]

$$\mathbb{L}_{\mathbb{X}}\mathcal{H} = \begin{pmatrix} \mathcal{L}_X g - (\mathcal{L}_X B - d\xi)g^{-1}B - B(\mathcal{L}_X g^{-1})B & (\mathcal{L}_X - d\xi)g^{-1} + B(\mathcal{L}_X g^{-1}) \\ - Bg^{-1}(\mathcal{L}_X B - d\xi) & \\ -g^{-1}(\mathcal{L}_X B - d\xi) - (\mathcal{L}_X g^{-1}B) & \mathcal{L}_X g^{-1} \end{pmatrix},$$

so that the T-duality conditions, eqs. (4.17) and (4.19), can be reformulated as the vanishing of the Dorfman derivative of $\mathcal{H}$,

$$\mathbb{L}_{\mathbb{X}}\mathcal{H} = 0; \tag{4.25}$$

in this sense, then, $\mathbb{X}$ is a generalised Killing vector of the generalised metric. We can now write a T-duality matrix [48],

$$\mathscr{T}_{\mathbb{X}} = 1 - 2\mathbb{X} \otimes \mathbb{X}^T \eta \in \mathrm{O}(n, n),$$

so that the T-dual generalised metric $\hat{\mathcal{H}}$, i.e. the metric given by acting with a T-duality transformation on $\mathcal{H}$ and corresponding to the dual theory defined by $\hat{g}$ and $\hat{B}$, is simply given by conjugation [26],

$$\mathcal{H} \xrightarrow{\text{T-duality}} \hat{\mathcal{H}} = \mathscr{T}_{\mathbb{X}}^T \mathcal{H} \mathscr{T}_{\mathbb{X}}.$$

In fact, in adapted coordinates, given frame fields e^i and duals e_i, we can even recast the T-duality matrix $\mathscr{T}_{\mathbb{X}}$ in the form of the factorised duality matrix in eq. (4.15).

4.6 Type II supergravity

So far, we have only dealt with a metric g, an antisymmetric (0,2)-tensor B, and (occasionally) a dilaton ϕ. We will now meet the remaining fields that populate type II supergravity. We will then compactify the theory. Actually, we will begin to do so by setting all the new fields – in fact, even the field strength of B – to vanish. We will then gradually re-introduce them and see how they change the resulting constraints on the geometry.

Type II supergravity is a 10-dimensional[52] theory with $\mathcal{N} = 2$ supersymmetries which emerges as the low-energy limit of type II string theory. Its two 16-dimensional[53] supersymmetry parameters are the Majorana-Weyl spinors $\epsilon^{1,2}$. That these must be Majorana-Weyl spinors follows from imposing supersymmetry at the level of the superstring action, which implies that the spinor coordinates $\theta^{1,2}$ – and so their corresponding supercharges, $\epsilon^{1,2} = \delta\theta^{1,2}$ – must be Majorana-Weyl. This is in analogy with super Yang-Mills (sYM) theories[54]. In closed superstring theory[55], we may choose $\theta^{1,2}$ to have opposite or equal chirality. This leads, respectively, to non-chiral and chiral theories. The corresponding supergravity theories are referred to as type IIA and type IIB supergravity, respectively. It follows that $\epsilon^{1,2}$ have opposite and equal chirality in type IIA and type IIB, respectively, so that the two theories have $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supersymmetries.

The Neveu-Schwarz-Neveu-Schwarz (NSNS) sector, consisting of those fields given by string states with NSNS boundary conditions, contains a metric g, an (antisymmetric) Kalb-Ramond B field with strength H, and a dilaton ϕ.

The matter content is made up of the fermionic doublets Ψ and λ – the gravitino and the dilatino, respectively. The gravitinos are of the same chirality as the associated supersymmetry parameters. We adopt the index convention whereby $a \in \{1,2\}$ labels doublets, while uppercase Roman letters, $M \in \{1,\ldots,10\}$, span

[52]Classically, we can discuss superstring theories also in 3, 4, and 6 dimensions, each of which carries different constraints on $\theta^{1,2}$. In the quantum theory, however, the critical dimension of superstring theory is 10 [49].

[53]In 10-dimensions, a spinor has 32 complex components. These are halved by the Weyl condition and made real by the Majorana condition, leading to Majorana-Weyl spinors having 16 real components [49].

[54]In d-dimensions, SYM actions are of the form

$$S = \int d^d x \left(-\frac{1}{4} F^2 + \frac{i}{2} \bar{\psi} \Gamma \cdot D\psi \right),$$

where

$$F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + g f^a{}_{bc} A^b_\mu A^c_\nu$$

is the non-Abelian field strength of the vector potential A^μ, ψ is a spinor field, and the Yang-Mills covariant derivative is

$$(D_\mu \psi)^a = \partial_\mu \psi^a + g f^a{}_{bc} A^b_\mu \psi^c.$$

In $d = 10$, A_μ carries $d - 2 = 8$ physical modes, so that supersymmetry imposes the existence of 8 fermionic physical modes; this singles out Majorana-Weyl spinors [49].

[55]For open strings, $\theta^{1,2}$ must have the same chirality, since they are matched at the end of the string. This leads to type I theories.

the 10-dimensions in which the theory lives. For instance, the metric components are labelled g_{MN}, while those of the gravitino doublet are Ψ^a_M.

The Ramond-Ramond (RR) sector consists of the p-form potentials C_p, where $p \in \{1,3,5,7,9\}$ for type IIA and $p \in \{0,2,4,6,8\}$ for type IIB. These can be assembled into the formal sums

$$C^- = C_1 + C_3 + C_5 + C_7 + C_9$$

for type IIA and

$$C^+ = C_0 + C_2 + C_4 + C_6 + C_8$$

for type IIB, which are Clifford(d,d) spinors of negative and positive chirality, respectively [6]. These forms have gauge transformations [35]

$$C_p \to C_p + d\kappa_{p-1} - H \wedge \kappa_{p-3}$$

for k-forms κ_k, and field strengths locally given by [50]

$$F_p = dC_{p-1} - H \wedge C_{p-3}, \tag{4.26}$$

where we ignore a subtlety[56] related to F_0.

We briefly pause to remark that, in the presentation above, we tacitly subscribed to the "democratic" formalism, for which there exists a duality relation

$$F_p = (-1)^{\lfloor p/2 \rfloor} \star_{10} F_{10-p}, \tag{4.27}$$

where $\lfloor\ \rfloor$ is the floor notation, and $\star_d$ is the d-dimensional Hodge star operator [12],

$$\star_d\, \omega = \frac{1}{k!(d-k)!}\sqrt{|g|}\varepsilon_{i_1 \dots i_d}\omega^{i_{d-k+1}\dots i_d} dy^{i_1} \wedge \dots \wedge dy^{i_{d-k}}$$

for some k-form ω. Due to this relation, F_6, F_7, F_8, F_9, and F_{10} are sometimes said to inhabit the "RR$\star$" sector. A (popular) alternative to this democratic formalism is to only consider F_0, F_2, and F_4 in type IIA, and F_1, F_3, and F_5 in type IIB, and ignore

[56]It can be shown that the Bianchi identities imply $F_0 = m$ for some constant m, the Romans mass [12]. Here we will focus on "Romans massless" type IIA theories, for which $m = 0$.

all other form fields. The sole vestige of the duality condition in eq. (4.27) in this second formalism is then the self-duality $F_5 = \star F_5$. In our democratic formulation, the redundancy introduced by considering a larger set of forms is compensated by the duality eq. (4.27); however, this must be imposed on top of the equations of motions, which is why the corresponding action is sometimes referred to as a "pseudo-action".

In terms of the formal sums of the field strengths

$$F^{+} = F_0 + F_2 + F_4 + F_6 + F_8 + F_{10}$$

in type IIA and

$$F^{-} = F_1 + F_3 + F_5 + F_7 + F_9$$

in type IIB, we can rewrite eq. (4.26) as[57]

$$F = d_H C,$$

where the twisted de Rham operator is $d_H = d - H\wedge$, as in eq. (3.16).

4.7 Fluxless supersymmetric compactifications

String theory at low energies is governed by the action of 10-dimensional supergravity,

$$S = \int_{M_{10}} \sqrt{-g}\left[\mathrm{e}^{-2\phi}\left(\mathcal{R} + 4(\partial\phi)^2 + \frac{1}{12}H^2\right) - \frac{1}{4}\sum_n \frac{1}{n!}\left(F_n\right)^2\right],$$

where $\mathcal{R}$ is the Ricci scalar of the metric g, and M_{10} is the ten-dimensonal spacetime. The equations of motion that follow from extremising the action above are

$$\mathcal{R}_{\mu\nu} - \frac{1}{4}H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}\mathrm{e}^{2\phi}\sum_n \frac{1}{(n-1)!}\left(F_n\right)_{\mu\lambda_1\ldots\lambda_{n-1}}\left(F_n\right)_{\nu}^{\lambda_1\ldots\lambda_{n-1}} = 0, \tag{4.28a}$$

$$\nabla^\lambda\left(\mathrm{e}^{-2\phi}H_{\mu\nu\rho}\right) - \frac{1}{2}\sum_n \frac{1}{(n-2)!}\left(F_n\right)_{\mu\nu\lambda_1\ldots\lambda_{n-2}}\left(F_{n-2}\right)^{\lambda_1\ldots\lambda_{n-2}} = 0, \tag{4.28b}$$

[57]For Romans massive type IIA theories, this reads $F = d_H C + F_0 e^{-B}$. In the case $m \neq 0$, eq. (4.26) is also modified.

$$\nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}\mathcal{R} - \frac{1}{48}H^2 = 0, \tag{4.28c}$$

$$\text{and} \quad \mathrm{d}F - H \wedge F = 0. \tag{4.28d}$$

A solution to these equations is referred to as a *background*. In fact, we will not attempt to solve these equations of motion. Rather, we will make a few key assumptions and recast eq. (4.28) into the language of generalised geometry.

Our first major assumption is that the solutions are compactified, i.e. that the 10-dimensional space takes the form of the warped product

$$M_{10} = M_4(y) \times_W X_6(x), \tag{4.29}$$

where M_4 is the external space, X_6 is a 6-dimensional curved, compact manifold (the internal space), and y and x are external and internal coordinates, respectively.

Solving eq. (4.28) is a daunting task. But it is also one we can evade, by conveniently assuming that the solutions are supersymmetric. It can be shown that the conditions for supersymmetry – which are significantly simpler than the above equations – in fact imply the equations of motion, so that supersymmetric backgrounds automatically solve eq. (4.28). To be more precise, this statement is only true if we supplement the supersymmetry conditions with the Bianchi identities for the supergravity fluxes.

In the case of global $\mathcal{N} = 1$ supersymmetry, for instance, the Hamiltonian is given by $H = Q^\dagger Q$, Q being a supercharge. A supersymmetric state Φ satisfies $Q\Phi = 0$, and so it is also a zero-energy solution to the full equations of motion, $H\Phi = Q^\dagger Q\Phi = 0$ [51].

4.7.1 A bosonic intermezzo

Consider a metric g on some d-dimensional manifold M with coordinates $\{y^\mu\}$. Under a flow generated by a vector field V, the infinitesimal change in the metric's components is

$$\delta g_{\mu\nu} = (\mathcal{L}_V g)_{\mu\nu} = 2\nabla_{(\mu}V_{\nu)},$$

where ∇ is the Levi-Civita covariant derivative, and $V_\nu = g_{\mu\nu}V^\mu$. Taking M to be Minkowski, so that $g_{\mu\nu} = \eta_{\mu\nu}$, the condition for V to be a Killing vector field, i.e. for the flow to be an isometry, now reads

$$\partial_{(\mu}V_{\nu)} = 0, \tag{4.30}$$

and is solved either by $V_\mu =$ constant, which corresponds to spacetime translations $y^\mu \to y^\mu + V^\mu$, or by $V_\mu = \Lambda_{\mu\nu}x^\nu$, where $\Lambda_{\mu\nu} = \Lambda_{[\mu\nu]}$, which corresponds to Lorentz transformations[58] $\delta x^\mu = \Lambda^\mu{}_\nu x^\nu = V^\mu$, where $\Lambda^\mu{}_\nu = \eta^{\mu\rho}\Lambda_{\rho\nu}$. In a background with multiple fields, we require

$$\mathcal{L}_V(\text{field}) = 0 \qquad \forall \text{ fields.}$$

We see that the choice of a Minkowski background breaks the symmetry group of the theory down to the Poincaré group. Furthermore, the existence of a Killing vector field V places topological and differential constraints on the manifold it inhabits. For instance, on a Riemannian manifold with $\mathrm{O}(d)$ structure, the existence of the globally defined, nowhere vanishing vector field V results in a reduction of the structure group to $\mathrm{O}(d-1)$ – the stabiliser of V. Topologically, the existence of such a vector field is equivalent[59] to the vanishing of the Euler characteristic, $\chi(M) = 0$, for a compact and orientible manifold M[60]. For other manifolds, the existence of such a field bears no topological implications. Furthermore, the differential condition contained in eq. (4.30) implies that the vector field must be a singlet of the holonomy group; this produces a corresponding reduction in the holonomy group.

Having briefly discussed the topological and differential conditions originating from the existence of a covariantly constant, globally defined, nowhere vanishing vector field, we are now ready to discuss their fermionic analogue – the supersymmetry conditions.

[58] To see that $V_\mu = \Lambda_{\mu\nu}y^\nu$ solves the Killing equation, take the partial derivative of both sides and symmetrise.

[59] One of the two directions of this equivalence follows from the Poincaré–Hopf index theorem.

[60] see for instance [52].

4.7.2 Type II fluxless compactifications

Let us return to type II supergravity, in all its fermionic glory. Consider a 10-dimensional space M_{10} given by the warped product in eq. (4.29). The most general 10-dimensional metric g that is compatible with this compactification ansatz has components

$$g_{MN}(y,x) = e^{2\lambda(x)} h_{\mu\nu}(y) + g_{mn}(x) \tag{4.31}$$

where $m, n \in \{1, \ldots, 6\}$ are the internal indices, $\mu, \nu \in \{0, \ldots, 3\}$ are the external indices, $M, N = (\mu, m)$ are the 10-d indices, $\lambda(x)$ is the warp factor, and $h_{\mu\nu}$ and g_{mn} are the metrics on the external and internal spaces, respectively. Note that the scale of the external space, namely the warp factor λ, is allowed to depend on the compact space.

In principle, we could consider a more general ansatz than that in eqs. (4.29) and (4.31), with off-diagonal terms in the metric (i.e. non-zero components carrying both external and internal indices). However, these would be vectors in M_4; the search for a supergravity *vacuum* leads us to require maximal symmetry, which in turn implies that any such off-diagonal term must vanish. We will therefore proceed with our original ansatz in eqs. (4.29) and (4.31).

As discussed earlier, the metric (together with the natural volume form) gives M_{10} an $\mathrm{SO}(9,1)$ structure; in fact, due to the compactification ansatz in eq. (4.29), the structure group is reduced to $\mathrm{SO}(3,1) \times \mathrm{SO}(6)$. Furthermore, we assume there exists a lift of the frame bundle – the principal $\mathrm{SO}(6)$ bundle associated to the tangent bundle of X_6 – to the universal cover $\mathrm{Spin}(3,1) \times \mathrm{Spin}(6)$; it will be helpful to recall the exceptional isomorphisms[61] $\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C})$ and $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$. It should be noted that that of the existence of a lift from $\mathrm{SO}(3,1) \times \mathrm{SO}(6)$ to its double cover is not a trivial assumption. Manifolds on which it is possible to define a spin bundle are referred to as spin manifolds, and the question on the nature of the topological conditions that allow for the existence of such a lift is an interesting one in its own right, which we will not explore further here.

We will now impose that the vacuum in eq. (4.29) preserve (at least some) supersymmetry, meaning that the variations of the background fields under (at least

[61] These accidental isomorphisms are only present for low-dimensional spin groups: $\mathrm{Spin}(n)$ has no such isomorphisms for $n > 6$.

some of) the supersymmetry generators vanish.

Firstly, consider a 10-dimensional spinor transforming in the $\mathbf{16}_R$ representation[62] of $SO(9,1)$. This decomposes under $SO(3,1) \times SO(6)$ as[63]

$$\mathbf{16}_R = (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}).$$

In terms of the universal cover of this decomposition, the lack of a $\mathrm{Spin}(3,1)$ singlet implies that a non-zero spinorial vacuum expectation value would break Poincaré invariance in the four uncompactified dimensions; this would violate the conditions for the solution to be a vacuum, and so we must set all spinorial fields to have vanishing vacuum expectation values.

It can be shown that the supersymmetric variations of bosonic fields obey

$$\delta_\epsilon(\text{bosonic field}) \sim (\text{fermionic field}),$$

where ϵ is the generator of the supersymmetry; above, we argued that fermionic fields should have vanishing vacuum expectation value, and so we see that the bosonic fields already satisfy the condition that the background be supersymmetric, namely the vanishing of the supersymmetric field variations. On the other hand, the supersymmetry variations of the gravitino and of the dilatino in the string frame can be shown to be [23]

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{4} \not{H}_M \mathcal{P} \epsilon + \frac{1}{16} e^\phi \sum_n \not{F}_n \Gamma_M \mathcal{P}_n \epsilon \tag{4.32a}$$

$$\text{and} \quad \delta\lambda = \left(\not{\partial}\phi + \frac{1}{2} \not{H} \mathcal{P} \right) \epsilon + \frac{1}{8} e^\phi \sum_n (-1)^n (5-n) \not{F}_n \mathcal{P}_n \epsilon, \tag{4.32b}$$

where

$$\mathcal{P} = \Gamma_{10} \quad \text{and} \quad \mathcal{P}_n = \Gamma_{10}^{n/2} \sigma^1$$

[62]Representations of $SO(9,1)$ can be obtained by analytically continuing those of $SO(10)$ [53].

[63]The notation used here follows from the fact that the fundamental representation $\mathbf{4}$ and its complex conjugate $\bar{\mathbf{4}}$ of $SU(4)$ are the spinorial representations of $SO(6)$. Indeed, a (real) spinor on a 6-dimensional manifold has eight components, and so decomposes into irreducible representations of $SU(4)$ as $\mathbf{8} = \mathbf{4} \oplus \bar{\mathbf{4}}$.

for type IIA, and

$$\mathcal{P} = -\sigma^3 \quad \text{and} \quad \mathcal{P}_n = \begin{cases} \sigma^1 & \text{for } (n+1)/2 \text{ even} \\ i\sigma^2 & \text{for } (n+1)/2 \text{ odd} \end{cases}$$

for type IIB, and we recall the slash notation introduced in eq. (3.25), so that, for instance,

$$\not{H}_M = \frac{1}{2} H_{MNP} \Gamma^{NP}.$$

In the absence of fluxes, eq. (4.32a) greatly simplifies, to give, reinstating the index labelling the doublet components,

$$\delta\psi_M^\alpha = \nabla_M \epsilon^\alpha. \tag{4.33}$$

The condition for the background to be supersymmetric therefore reads

$$\nabla_M \epsilon^\alpha = 0 \tag{4.34}$$

– a Killing spinor equation.

By focussing on the spacetime component of eq. (4.34), we obtain the following integrability condition, [51]

$$\nabla_\mu \epsilon + \frac{1}{2}(\gamma_\mu \gamma_5 \otimes \not{\nabla}\lambda)\epsilon = 0 \qquad \Rightarrow \qquad [\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{2}\nabla_m \lambda \nabla^m \lambda \gamma_{\mu\nu}\epsilon, \tag{4.35}$$

where ∇ and γ_μ are the covariant derivative and gamma matrices associated with the metric h. We now note for later reference that the 10-dimensional gamma matrices are given by [23]

$$\Gamma^M = (\Gamma^\mu, \Gamma^m) = (\gamma^\mu \otimes 1, \gamma^5 \otimes \gamma^m),$$

where $\gamma_5 = \frac{i}{4!}\varepsilon_{\sigma\rho\mu\nu}\gamma^{\sigma\rho\mu\nu}$, and we also define the antisymmetric product

$$\Gamma^{M\ldots N} = \Gamma^{[M} \ldots \Gamma^{N]}.$$

The 10-dimensional covariant derivative of a spinor then takes the form

$$\nabla_M \epsilon = \partial_M \epsilon + \frac{1}{4}\omega_{MAB}\Gamma^{AB}\epsilon,$$

where ω_{MAB} are the components of the spin connection and we introduced a vielbein (in particular, a zehnbein), the flat indices being A, B [51].

We also have that [23]

$$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\epsilon,$$

which – under the assumption of maximal symmetry on M_4,

$$R_{\mu\nu\rho\sigma} = k(h_{\mu\rho}h_{\nu\sigma} - h_{\mu\sigma}h_{\nu\rho})$$

– becomes

$$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{k}{2}\gamma_{\mu\nu}\epsilon.$$

Comparing this with eq. (4.35) yields the integrability condition

$$\nabla_m\lambda\nabla^m\lambda = -k.$$

Recall λ is only allowed to depend on the internal coordinates, and so, taking the internal space to be compact, it must have a maximum. $\nabla_m\lambda\nabla^m\lambda$ being constant then implies that

$$k = 0 \quad \text{and} \quad \nabla_m\lambda = 0,$$

so that M_4 must be Minkowski $\mathbb{R}^{3,1}$ – the other maximally symmetric solutions, namely anti-de Sitter and de Sitter spacetimes, have non-zero scalar curvature (< 0 for AdS, > 0 for dS) [54]. We will later see that the presence of fluxes allows us to relax this condition and therefore to discuss proper warped compactifications. For the remainder of this section, we will take the warp factor to vanish and $h_{\mu\nu}$ to be the Minkowski metric.

Under the decomposition Clifford$(9,1) \to$ Clifford$(3,1) \times$ Clifford(6) that follows from the compactification ansatz, we can separate the supersymmetry generators into external and internal spinors as

$$\epsilon^1_{\text{IIA}} = \xi^1_+ \otimes \eta^1_- + \xi^1_- \otimes \eta^1_+ \tag{4.36a}$$

$$\epsilon^2_{\text{IIA}} = \xi^2_+ \otimes \eta^2_+ + \xi^2_- \otimes \eta^2_- \tag{4.36b}$$

for type IIA, and

$$\epsilon^{1,2}_{\mathrm{IIB}} = \xi^{1,2}_{+} \otimes \eta^{1,2}_{-} + \xi^{1,2}_{-} \otimes \eta^{1,2}_{+} \tag{4.36c}$$

for type IIB. The above decomposition automatically satisfies the Weyl condition (given the $\pm$ subscripts indicate $\pm$-chirality), and is also Majorana if we attach the further conditions

$$\xi_{-} = (\xi_{+})^{*} \quad \text{and} \quad \eta_{-} = (\eta_{+})^{*}.$$

A phenomenologically interesting question is the number of unbroken supersymmetries which descend to the four-dimensional space. For example, in the case of a field ψ with supersymmetric transformation $\delta\psi = \nabla\epsilon$, with ϵ being the gauge parameter, unbroken supersymmetric charges in the four-dimensional space correspond to solutions to $\nabla\epsilon = 0$ in the internal space [55]. The decomposition in eq. (4.36) identifies an $\mathcal{N} = 2$ effective theory in four dimensions, with the eight supercharges parametrised by (ξ^1, ξ^2) [7]. From a phenomenological perspective, this amount of unbroken supersymmetry is prohibitively large: the algebra is such that left- and right-handed Fermi fields have equal gauge transformations. Realistic particle physics models therefore require $\mathcal{N} < 2$; later, we will see that the introduction of fluxes along the compact directions allows us to relate $\epsilon^{1,2}$ and thus generically lead to minimal $\mathcal{N} = 1$ in four dimensions.

Under the decomposition in eq. (4.36), the supersymmetry condition for the gravitino field, eq. (4.34), separates into conditions on the internal and external spaces:

$$\partial_\mu \xi_{\pm} = 0, \tag{4.37a}$$

$$\text{and} \quad \nabla_m \eta_{\pm} = 0. \tag{4.37b}$$

The integrability condition on the internal manifold X_6 yields

$$\nabla_m \eta = 0 \quad \Rightarrow \quad [\nabla_m, \nabla_n]\eta = 0 \quad \Rightarrow \quad R_{mnpq}\Gamma^{pq}\eta = 0 \quad \Rightarrow \quad R_{mn} = 0. \tag{4.38}$$

This condition then implies that the internal manifold must be Ricci-flat [51].

Equation (4.37b) is a Killing spinor equation. It establishes both topological and differential constraints on the internal manifold X_6, which we will now review.

The most primitive condition that follows from eq. (4.37b) is the existence of globally defined, non-vanishing sections $\eta_+^{1,2}$ of the spinor bundle[64]. This is needed in order to be able to decompose the modes of the 10-d fields into $\mathcal{N} = 2$ multiplets, and it translates into a condition on the topology of the internal manifold, as we will now see. Firstly, let us assume that $\eta_+^1 = \eta_+^2 \equiv \eta_+$, so that we only have one internal spinor (and its complex conjugate). We will later relax this assumption and reinstate the second internal spinor. Taking a basis in which

$$\eta_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \end{pmatrix}, \tag{4.39}$$

we see that the subgroup of SU(4) under which η_+ is invariant is made up of elements of the form

$$S = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{where} \qquad A \in \mathrm{SU}(3) \subset \mathrm{SU}(4),$$

meaning that the transition functions must be in SU(3) for the global nature of η_+ to truly be such. We see that the introduction of a global, non-degenerate tensor field – or in this case, the spinor field η_+ – entails a reduction in the structure group, as the different descriptions of the field in various patches must agree on the overlaps between patches. This yields the topological condition resulting from the presence of supersymmetry: the structure group is reduced from SU(4) to SU(3).

The fact that a global section η_+ not only exists, but is also covariantly constant, as in eq. (4.37b), imposes a further, differential condition on the internal manifold. For it to be covariantly constant, the spinor must be a singlet of the holonomy group, i.e. it must return to itself upon being parallely transported along any loop. Consequently, the holonomy group also reduces from Spin(6) $\simeq$ SU(4) to SU(3) – this can be seen by retracing the steps above, namely placing η_+ in the form of eq. (4.39) by means of an SU(4) transformation, and then noting that its stabiliser is SU(3).

Table 4.1 contains the reduced holonomy groups for various dimensions.

This reduction can also be observed at the level of representations, by demand-

[64]Spinor fields are sections of spinor bundles.

dim(M)	general holonomy		reduced holonomy	geometry
4	SO(4)	$\supseteq$	SU(2)	Calabi-Yau
6	SO(6)	$\supseteq$	SU(3)	Calabi-Yau
7	SO(7)	$\supseteq$	G_2	Exceptional
8	SO(8)	$\supseteq$	Spin(7)	Exceptional

Table 4.1: The reduction in the holonomy group on various manifolds due to the existence of covariantly constant spinor fields.

ing that spinors remain uncharged upon being parallelly transported. Spinors with definite chirality living on 6-dimensional spaces inhabit either the $\mathbf{4}$ or the $\overline{\mathbf{4}}$ irreducible representations of the general SU(4) holonomy group, neither of which are singlets. On the other hand, a reduction of the holonomy group to the subgroup SU(3) implies the decompositions

$$\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$$

and

$$\overline{\mathbf{4}} = \overline{\mathbf{3}} \oplus \mathbf{1}$$

of the spinorial representations, where the newfound singlets enable the existence of covariantly constant spinors. Furthermore, SU(3) is the largest subgroup of SU(4) which contains such holonomy singlets [51].

We have already studied SU(3) structures and holonomies, albeit in the language of differential forms and algebraic structures. We now recast the topological and differential conditions following from supersymmetry into the language familiar from chapter 3, as to connect the two formulations.

Given a covariantly constant spinor η_+, we can assign it a unitary normalisation, $\eta_+^\dagger \eta_+ = 1$, by applying the Leibnitz identity[65] to eq. (4.37b). We can then build a real 2-form ω and a complex 3-form Ω, which inherit from η_+ the properties of being globally defined and everywhere non-vanishing. Explicitly, the construction is [54]

$$\omega_{mn} = -i\eta_\pm^\dagger \gamma_{mn} \gamma \eta_\pm, \tag{4.40}$$

[65]Note that $\eta_+ \dagger \eta_+$ is a scalar in Euclidean signature.

where γ is the chirality operator (and so $\gamma\eta_+ = \eta_+$), and

$$\Omega_{mnp} = -i\eta_-^\dagger \gamma_{mnp}\eta_+. \tag{4.41}$$

By means of the Fierz identities, it can be shown that

$$J^m_{\ n} = g^{mp}\omega_{pn}$$

satisfies

$$J^m_{\ n} J^n_{\ p} = -\delta^m_p.$$

We have found an almost complex structure [54]. Furthermore, the metric is Hermitian with respect to it,

$$J^m_{\ n} J^p_{\ q} g_{mp} = g_{nq}.$$

In fact, we have that

$$\nabla_p J_m^{\ n} = \nabla_p(-2i\eta_+^\dagger \gamma_m^{\ n}\eta_+);$$

the metric is covariantly constant, and $\nabla_m\eta_\pm = 0$ is exactly the Killing spinor equation, eq. (4.37b), so that [51]

$$\nabla_p J_m^{\ n} = 0,$$

which implies the vanishing of the Nijenhuis tensor, $N^p_{\ mn} = 0.$. We see that, in this picture, the supersymmetry conditions take the form of the integrability of the almost complex structure J: the internal space is a complex manifold equipped with a Hermitian metric. In terms of the forms in eqs. (4.40) and (4.41), we find the non-degeneracy and compatibility conditions

$$\omega \wedge \omega \wedge \omega \sim \Omega \wedge \overline{\Omega} \neq 0, \tag{4.42a}$$

$$\text{and} \qquad \omega \wedge \Omega = 0, \tag{4.42b}$$

with Ω also being decomposable. Furthermore, it follows that ω is closed,

$$d\omega = 0,$$

so that the internal space is Kähler [51]. In fact, Ω is also closed, $d\Omega = 0$. These are the differential requirements that follow from supersymmetry.

Regardless of which description we choose, we find that the internal manifold must be Calabi-Yau. We see that the supersymmetry conditions translate into remarkably stringent constraints on the geometry of the internal space.

We briefly digress to notice that the Calabi-Yau description in terms of ω and Ω arises naturally at the level of representations. As mentioned earlier, the spinorial representation of SO(6) decomposes into SU(3) representations as $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$, where the singlet corresponds to the globally defined, nowhere vanishing spinor. The vector, 2-form and 3-form representations of SO(6) similarly decompose into

$$\mathbf{6} \to \mathbf{3} \oplus \bar{\mathbf{3}}, \tag{4.43a}$$

$$\mathbf{15} \to \mathbf{8} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}, \tag{4.43b}$$

$$\text{and} \qquad \mathbf{20} \to \mathbf{6} \oplus \bar{\mathbf{6}} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1} \oplus \mathbf{1}, \tag{4.43c}$$

respectively. We indeed see the emergence of both 2-form and 3-form singlets; as anticipated, these are the SU(3)-invariant ω and Ω forms defined above from η. We also note the lack of vector – and therefore, in six dimensions, 5-form – singlets, from which the compatibility condition in eq. (4.42b) immediately follows. Similarly, the singlet nature of six-forms gives eq. (4.42a) [56].

4.7.3 Effective theory

We now turn to the issue of the effective description of fluxless type II compactifications at low energies. We begin by recalling that Hodge's decomposition theorem on compact Riemannian manifolds leads to the following isomorphism,

$$\mathrm{Harm}^n = H^n,$$

where Harm^n is the space of n-forms τ such that $\Delta\tau = 0$,

$$\Delta = dd^\dagger + d^\dagger d = (d + d^\dagger)^2$$

being the operator measuring four-dimensional masses. The above isomorphisms implies that there is a unique harmonic representative for each cohomology class, meaning we can take the bases presented in eq. (2.21) to consist of harmonic forms [22]. Recall that on a Calabi-Yau manifold, there exist a harmonic 0-form $r_0(x) \equiv 1$, the (3,0)-form Ω (and so a (0,3)-form $\overline{\Omega}$), the (3,3)- volume form, $h^{1,1}$ harmonic (1,1)- and (2,2)-forms, and finally $h^{2,1}$ (2,1)- and (1,2)-forms. At low energies, we can expand the fields living in 10-dimensions in this basis of harmonic forms.

We begin by expanding the dilaton ϕ as

$$\phi(y, x) = r_0(x)\phi(y), \tag{4.44}$$

where recall the only internal scalar $r_0(x) = 1$. The four-dimensional field $\phi(y)$ is a modulus, i.e. a massless scalar. The lack of harmonic one-forms implies that the expansion of the B-field has no components with one external and one internal indices. The purely internal component is decomposed in $\{r_a\}$ – in fact, this is invariant under the gauge transformations of B, since gauge fields are (rather tautologically) only defined up to cohomology. Hence,

$$B(y, x) = r_0(x)B(y) + B^a(y)r_a(x), \tag{4.45}$$

where $B(y)$ and $B^a(y)$ are, once again, moduli. We quote from [56] the remaining expansions, namely that of the metric

$$g_{i\bar{\jmath}}(x, y) = iv^a(x)\,(\omega_a)_{i\bar{\jmath}}\,(y) \quad \text{and} \quad g_{ij}(x, y) = i\bar{z}^k(x)\left(\frac{(\bar{\chi}_k)_{i\bar{k}\bar{l}}\,\Omega^{\bar{k}\bar{l}}_{j}}{|\Omega|^2}\right)(y), \tag{4.46}$$

and those of the Ramond-Ramond potentials,

$$C_1(y, x) = r_0(x)C_1^0(y) \tag{4.47a}$$

$$\text{and} \quad C_3(y, x) = C_1^a(y)r_a(x) + \xi^K(y)\alpha_K(x) - \tilde{\xi}_K(y)\tilde{\alpha}^K(x) \tag{4.47b}$$

for type IIA, and

$$C_0(y, x) = r_0(x)C_0(y), \tag{4.47c}$$

$$C_2(y, x) = r_0(x)C_2(y) + c^a(y)r_a(x) \tag{4.47d}$$

$$\text{and} \quad C_4(y,x) = V_1^K(y)\alpha_K(x) + \rho_a(y)\tilde{r}^a(x) \tag{4.47e}$$

for type IIB.

Substituting these into the ten-dimensional actions, and integrating out the Calabi-Yau, we find the four-dimensional effective actions [56]

$$\begin{aligned} S^{\text{eff}}_{\text{IIA}} = \int_{\mathbb{R}^{3,1}} &-\frac{1}{2}R\star 1 + \frac{1}{2}\operatorname{Re}\mathcal{N}_{AB}F^A\wedge F^B + \frac{1}{2}\operatorname{Im}\mathcal{N}_{AB}F^A\wedge\star F^B \\ &- G_{ab}dt^a\wedge\star d\bar{t}^b - h_{uv}dq^u\wedge\star d\bar{q}^v \end{aligned}$$

for type IIA, and analogously for type IIB,

$$\begin{aligned} S^{\text{eff}}_{\text{IIB}} = \int_{\mathbb{R}^{3,1}} -\frac{1}{2}R\star 1 + \frac{1}{2}\operatorname{Re}\mathcal{M}_{KL}F^K\wedge F^L + \frac{1}{2}\operatorname{Im}\mathcal{M}_{KL}F^K\wedge\star F^L \\ -G_{k\ell}dz^k\wedge\star d\bar{z}^\ell - h_{pq}d\tilde{q}^p\wedge\star d\tilde{q}^q. \end{aligned}$$

Let us dissect $S^{\text{eff}}_{\text{IIA}}$. The analysis for type IIB is analogous. Firstly, $\int R\star 1$ is the usual gravitational term. Next, we have a gauge kinetic term featuring the field strengths $F^A = (dC_1^0, dC_1^a)$. The coupling matrix $\mathcal{N}$ can be shown to be given by

$$\begin{aligned} \operatorname{Re}\mathcal{N} &= \begin{pmatrix} -\frac{1}{3}\mathcal{K}_{abc}B^aB^bB^c & \frac{1}{2}\mathcal{K}_{abc}B^bB^c \\ \frac{1}{2}\mathcal{K}_{abc}B^bB^c & -\mathcal{K}_{abc}B^c \end{pmatrix}, \\ \text{and} \quad \operatorname{Im}\mathcal{N} &= -\frac{\mathcal{K}}{6}\begin{pmatrix} 1+4G_{ab}B^aB^b & -4G_{ab}B^b \\ -4G_{ab}B^b & 4G_{ab} \end{pmatrix}. \end{aligned}$$

The main focus of our analysis, however, will be the final term, which contains the complex scalar fields t (in type IIA) and z (in IIB), as well as the metrics G_{ab} and $G_{k\ell}$, which set the relative scaling of the kinetic terms. As we will see, all these terms arise from purely geometrical considerations.

The action $S^{\text{eff}}_{\text{IIA}}$ is that of a $\mathcal{N}=2$ ungauged supergravity theory in four dimensions. In such a theory, the moduli we identified in eqs. (4.44) to (4.47) fall into $\mathcal{N}=2$ multiplets. These are collected in table 4.2. In particular, we will now focus on the scalars in the vector multiplets of type IIA and type IIB and on the spaces they span. The scalars contained in the hypermultiplets, on the other hand, are related to a quaternionic manifold; we will not consider these further here.

type IIA	gravity multiplet	$(g_{\mu\nu}, C_1^0)$
	$h^{1,1}$ vector multiplets	(C_1^a, v^a, B^a)
	$h^{2,1}$ hypermultiplets	$(z^k, \xi^k, \tilde{\xi}_k)$
	tensor multiplet	$(B_{\mu\nu}, \phi, \xi^0, \tilde{\xi}_0)$
type IIB	gravity multiplet	$(g_{\mu\nu}, V_1^0)$
	$h^{2,1}$ vector multiplets	(V_1^k, z^k)
	$h^{1,1}$ hypermultiplets	(v^a, B^a, c^a, ρ_a)
	tensor multiplet	$(B_{\mu\nu}, C_2, \phi, C_0)$

Table 4.2: The $\mathcal{N} = 2$ multiplets in type IIA (above) and IIB (below) and their moduli content.

Recall that our vacuum has a cross product structure $\mathbb{R}^{3,1} \times X_6$, so that the internal manifold X_6 – whose geometry, recall, is defined by parameters ω and Ω – is taken to be the same at every point in our Minkowski spacetime. In some sense, we can view this as a "vacuum expectation value" for the geometry of the effective field theory. Upon perturbing the fields from their background values, however, we allow the internal manifold to vary across $\mathbb{R}^{3,1}$, effectively turning the scalars ω and Ω into functions of the external coordinates, $\omega(y)$ and $\Omega(y)$. This is depicted in fig. 4.1.

The space of possible ω, the Kähler moduli, turns out to be precisely the cohomology group $H^{1,1}$. In terms of the usual basis $\{r_a\}$, then, we can expand

$$\omega(y) = \omega_0 + t^a(y) r_a,$$

where ω_0 is the value of ω in the vacuum, i.e. what we previously referred to as ω. We see the emergence of the scalars t_a as parametrising the pertubation in the geometry-defining parameter ω. These complex combinations of Kähler and B-field deformations are sometimes referred to as complexified Kähler deformations. In terms of the expansions of the metric and the B-field, we have that $t^a = B^a + iv^a$.

The metric G_{ab} also has a geometrical origin:

$$G_{ab} = \frac{1}{4\text{vol}} \int_{X_6} r_a \wedge \star r_b,$$

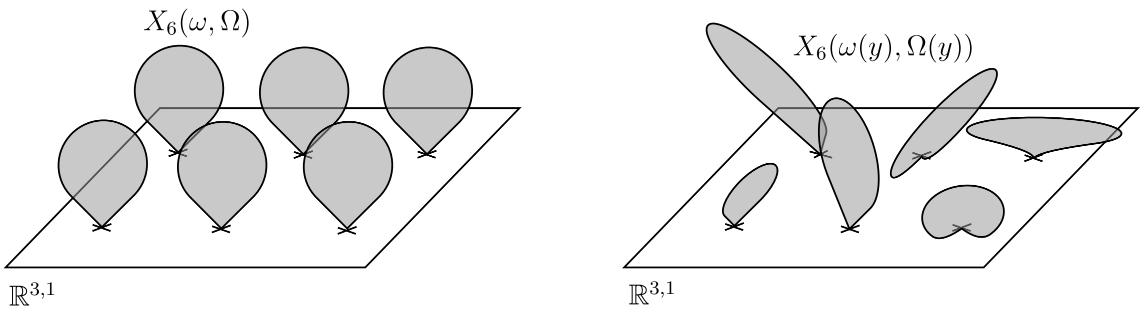


Figure 4.1: (Left.) A "vacuum expectation value" for the geometry, with the base space being Minkowski, $\mathbb{R}^{3,1}$, and the same manifold X_6 (represented by a grey blob), defined by ω and Ω, at every point in the base manifold. (Right.) In this context, a perturbation effectively implies allowing the internal manifold X_6 to vary across $\mathbb{R}^{3,1}$. The size and shape of the deformations are purely illustrative.

so that it is (related to) the intersection number associated with r_a and $\star r_b$.

A similar story develops for the scalars z^k. In particular, their nature is that of deformations of the complex structure. This is due to the fact that the complex structure moduli, i.e. the space of possible Ω, is the cohomology group $H^{2,1}$. Choosing a (complex) basis $\{\chi_k\}$, we can express perturbations about the background value Ω_0 of Ω as

$$\Omega(y) = \Omega_0 + z^k(y)\chi_k.$$

The metric $G_{k\ell}$ also has a geometrical origin [56],

$$G_{k\ell} = \frac{-1}{\int \Omega \wedge \overline{\Omega}} \int_{X_6} \chi_k \wedge \overline{\chi}_\ell.$$

4.8 Flux compactifications

We will now relax the assumption that the NSNS and RR fluxes vanish. That of flux compactifications is a very active area of research: flux compactifications are attractive because they break the amount of preserved supersymmetry in a stable way, they provide a stabilisation mechanism for at least some of the moduli, and could explain the hierarchy problem, amongst other interesting prospects [13]. Furthermore, following the conjecture of the AdS/CFT correspondence [57], it was found that certain type IIB flux solutions could be dual to confining gauge theories

– see, for instance, [58].

4.8.1 $\mathbf{SU}(3) \times \mathbf{SU}(3)$ structures

Prior to turning back on the fluxes, let us consider the case of two internal spinors, η^1_+ and η^2_+ (and their complex conjugates, of course). As we have seen, each defines an SU(3) structure, which may be characterised in terms of a real 2-form ω and a complex 3-form Ω. Similarly, two SU(3) structures are given by two copies of (ω, Ω). Locally,

$$\omega^1 = j + v \wedge v', \tag{4.48a}$$

$$\omega^2 = j - v \wedge v', \tag{4.48b}$$

$$\Omega^1 = k \wedge (v + iv'), \tag{4.48c}$$

$$\text{and} \quad \Omega^2 = k \wedge (v - iv'), \tag{4.48d}$$

where v, v' are real 1-forms, j is a real (1,1)-form, and k is a complex (2,0)-form . Together, $\{j, k, v, v'\}$ form a local SU(2) structure. Indeed, the two SU(3) structures defined by η^1_+ and η^2_- intersect on an SU(2) structure on T, unless the two spinors are parallel everywhere, in which case they give rise to a single SU(3) structure on T [33]. Of course, these are merely the two ends of a spectrum of possibilities. We will now briefly formalise this analysis.

The most general relation[66] between the two spinors is

$$\eta^2_+ = c\eta^1_+ + d(v + iv')_m \gamma^m \eta^1_-, \tag{4.49}$$

for some complex functions c, d such that $|c|^2 + |d|^2 = 1$, with c [d] vanishing if $\eta^{1,2}_+$ are perpendicular [parallel]. It is convenient to introduce a spinor χ_+, normalised such that $\chi^\dagger_+ \chi_+ = 1$ and orthogonal to η^1_+, i.e. $\chi^\dagger_+ \eta^1_+ = 0$. We can then write

$$\eta^2_+ = \cos\varphi \eta^1_+ + \sin\varphi \chi_+,$$

where $0 \leq \varphi \leq \pi/2$ is the angle between η^1_+ and η^2_+. In the case of $\varphi = 0$ everywhere,

[66] A basis for Clifford(6) spinors is provided by $\{\eta^1_+, \gamma^m\eta^1_+, \gamma^m\eta^1_-, \eta^1_-\}$. However, since the left-hand side of eq. (4.49) has positive chirality, only η^1_+ and $\gamma^m\eta^1_-$ can be used.

i.e. for $\eta^1 \parallel \eta^2$, we need not define χ_+, and so we have a single SU(3) structure. If instead $\eta^1_+ \perp \eta^2_+$ everywhere, the resulting local SU(2) structure – which in fact turns into a global SU(2) structure in this case – is referred to as "static". The intermediate case in which η^1 and η^2 are generically neither parallel nor orthogonal, and φ varies along the manifold, is referred to as a "dynamic intermediate" SU(2) structure. The case in which η^1_+ and η^2_+ are never parallel comes with more stringent constraints – for instance, the complex vector obtained by the bilinear $\eta^{1\dagger}_+ \gamma_m \eta^2_-$ is nowhere vanishing, and so the Euler characteristic of the manifold must vanish [38].

The SU(3) and SU(2) structures on T are particular cases of a more general case, namely that of SU(3)×SU(3) structures on $T \oplus T^*$. These have a more natural description, one in terms of generalised objects, and in particular, Spin(6, 6) spinors. Given Spin(6) spinors $\eta^{1,2}_+$, recall we assembled two naked Spin(6, 6) pure spinors in eq. (3.24) as

$$\Phi^\pm = e^{-\phi} \eta^1_+ \otimes \overline{\eta}^2_\pm. \tag{4.50}$$

In the above, we introduced the dilaton. Its introduction in terms of the norm of the pure spinors [36] is related to the trivialisation of $\sqrt{\det T}$ in eq. (3.23) [32]; in this sense, then, the dilaton can be regarded as "defining" the polyform-spinor isomorphism [15].

We may also dress these with B-transformations as follows,

$$\Phi^\pm_D \equiv e^B \Phi^\pm, \tag{4.51}$$

where again we remark that $\Phi^\pm$ and $\Phi^\pm_D$ are sections of the spinor bundle over $T \oplus T^*$ and of that over the twisted generalised bundle, respectively.

In particular, as we mentioned in section 3.7, each pure spinor Φ defines an SU(3, 3) structure. For two pure spinors to define a common SU(3)×SU(3) structure, we need to impose certain compatibility conditions between Φ^+ and Φ^-, namely that they have equal normalisation, and that

$$\dim(L_{\Phi^+} \cap L_{\Phi^-}) = 3.$$

Rephrasing them in terms of the Mukai pairing in eq. (3.26), the normalisation and compatibility conditions become

$$
\begin{aligned}
& (\Phi^+, \overline{\Phi}^+) = (\Phi^-, \overline{\Phi}^-) && \text{(4.52a)} \\
\text{and} \quad & (\Phi^+, \mathbb{X} \cdot \Phi^-) = (\overline{\Phi}^+, \mathbb{X} \cdot \Phi^-) = 0, && \text{(4.52b)}
\end{aligned}
$$

the latter holding for all sections $\mathbb{X}$ [33]. Note that the spinors in eq. (4.51) do satisfy these conditions, and therefore define an SU(3)×SU(3) structure.

The positive- and negative-chirality spinor bundles $S^{\pm}$ are both 32-dimensional, giving a total of 64 dimensions. The compatibility equations in eq. (4.52) amount to 13 conditions. We can transform between structures by means of $\mathrm{O}(6,6)$ transformations. Bearing in mind that each such structure is invariant under SU(3)×SU(3), we see that the space of structures is given by

$$
\Sigma = \frac{\mathrm{O}(6,6)}{\mathrm{SU}(3) \times \mathrm{SU}(3)},
$$

which is 50-dimensional. A pair of compatible pure $\mathrm{Spin}(6,6)$ spinors therefore provides an embedding

$$
(\Phi^+, \Phi^-) : \Sigma \times \mathbb{R}^+ \hookrightarrow S^+ \oplus S^-
$$

where the $\mathbb{R}^+$ factor consists in rescalings of the spinors [7].

Making use of the local SU(2) structure presented in eqs. (4.48) and (4.49), we may write the naked spinors as (momentarily dropping the factor related to the dilaton ϕ)

$$
\begin{aligned}
& \Phi^+ = +(\bar{c} e^{-ij} - i\bar{d} k) \wedge e^{-iv \wedge v'} && \text{(4.53a)} \\
\text{and} \quad & \Phi^- = -(d e^{-ij} + ick) \wedge (v + iv'). && \text{(4.53b)}
\end{aligned}
$$

If $\eta^1 \parallel \eta^2$ at a point, then we find

$$
\begin{aligned}
& \Phi^+ = e^{-ij} \wedge e^{-iv \wedge v'} \\
\text{and} \quad & \Phi^- = -ik \wedge (v + iv'),
\end{aligned}
$$

	SU(3) structure	SU(2) structure	SU(3)×SU(3) structure
Φ^+	$e^{-i\omega}$	$-ik \wedge e^{-iv\wedge v'}$	$(\bar{c}e^{-ij} - i\bar{d}k) \wedge e^{-iv\wedge v'}$
Φ^-	$-i\Omega$	$-e^{-ij} \wedge (v + iv')$	$-(de^{-ij} + ick) \wedge (v + iv')$

where $\omega = j + v \wedge v'$ and $\Omega = k \wedge (v + iv')$

Table 4.3: The pure spinors $\Phi^{\pm}$ characterising SU(3) and SU(2) structures on T and SU(3)×SU(3) structures on $T \oplus T^*$ [33, 38].

which, in light of eq. (4.48), should be taken to mean

$$\Phi^+ = e^{-i\omega} \quad \text{and} \quad \Phi^- = -i\Omega. \tag{4.54}$$

This corresponds to the earlier case, in which two coinciding spinors define a single SU(3) structure. They are annihilated by [38]

$$(\imath_m - i\omega_{mn} dx^n \wedge) e^{-i\omega} = 0 \quad \text{and} \quad dx^m \wedge \Omega = \imath_{\overline{m}} \Omega = 0.$$

These correspond to the eigenbundles of the generalised almost complex structures in eq. (3.21).

Conversely, if $\eta^1 \perp \eta^2$, then

$$\begin{aligned} \Phi^+ &= -ik \wedge e^{-iv\wedge v'} \\ \text{and} \quad \Phi^- &= -e^{-ij} \wedge (v + iv'), \end{aligned}$$

where we notice that Φ^+ takes the form of $-i\Omega$ in four dimensions, and that of $e^{i\omega}$ in two-dimensions, and vice versa for Φ^-.

The results are summarised in table 4.3.

4.8.2 Vacua with Neveu-Schwarz-Neveu-Schwarz fluxes

We now consider the case of a non-vanishing NSNS flux $H = dB$.

As already mentioned, the conditions for the preservation of four-dimensional $\mathcal{N} = 2$ supersymmetry in the presence of a non-trivial H – but with the Ramond-Ramond fluxes again absent, at least for now – were found in [43]; they were later reformulated in the language of generalised complex geometry in [59]. In the untwisted picture (in which we consider the untwisted generalised tangent bundle, and explicitly twist the integrability conditions, amongst others), they read

$$\begin{aligned} d_H \Phi^+ &= 0, \\ \text{and} \quad d_H \Phi^- &= 0. \end{aligned}$$

Recall from section 3.7 that the H-twisted closure of two pure spinors, implied by the equations above, defines a generalised Calabi-Yau metric structure. We may also see this enhancement of supersymmetry at the level of the split in eq. (4.36) – in the absence of Ramond-Ramond fluxes relating ϵ^1 and ϵ^2, the most general solution is to take two different external spinors ξ^1 and ξ^2. This then leads to an $\mathcal{N} = 2$ solution [15, 38].

We also mention the results obtained in [9, 60–63] that the equations for $\mathcal{N} = 1$ supersymmetry with a non-trivial H flux are

$$\begin{aligned} d_H(e^{-\phi}\Phi^-) &= 0, \\ \text{and} \quad d(e^{-\phi}\Phi^+) &= ie^{-2\phi} \star H, \end{aligned}$$

where $\Phi^+ = e^{-\phi}e^{-i\omega}$ and $\Phi^- = -ie^{-\phi}\Omega$ – i.e. the pure spinors in eq. (4.54), with the dilaton factor reinstated. Interestingly, the twisting of the equation of motion of Φ^+ is not of the usual form [13].

4.8.3 $\mathcal{N} = 1$ vacua with Neveu-Schwarz-Neveu-Schwarz and Ramond-Ramond fluxes

In this section, we will analyse the description of $\mathcal{N} = 1$ vacua in terms of generalised structures. As before, we will find both algebraic and differential conditions – the former originating from requiring an $\mathcal{N} = 2$ effective theory in four dimensions, whilst the latter from demanding an $\mathcal{N} = 1$ vacuum [38].

In [38], it was found that the conditions for $\mathcal{N} = 1$ supersymmetric vacua can be given in terms of pure Clifford(6,6) spinors (see eqs. (4.51) and (4.53)) for vanishing cosmological constant as

$$d_H(e^{2\lambda-\phi}\Phi_1) = 0, \tag{4.57a}$$

$$\text{and} \quad d_H(e^{2\lambda-\phi}\Phi_2) = e^{2\lambda-\phi}d\lambda \wedge \overline{\Phi}_2 + \frac{i}{8}e^{3\lambda} \star \sigma(f), \tag{4.57b}$$

where f is a form on the internal manifold such that

$$F = f + \text{vol}_4 \wedge \tilde{f}. \tag{4.58}$$

The form of f above follows from the requirement that Poincaré invariance be preserved on M_4. The duality in eq. (4.27) appears here as

$$\tilde{f} = \sigma(\star_6 f).$$

In eq. (4.57), we used the σ map defined in eq. (3.27), and we introduced the notation

$$\Phi_1 = \begin{cases} \Phi^+ & \text{for type IIA} \\ \Phi^- & \text{for type IIB,} \end{cases}$$

and viceversa for Φ_2.

There is also a condition on the norm of the pure spinors that follows from $\mathcal{N} = 1$ supersymmetry; defining $|a|^2 = |\eta_1^+|^2$ and $|b|^2 = |\eta_2^+|^2$, so that

$$|\Phi_\pm|^2 = |a|^2|b|^2,$$

the condition reads [38]

$$d|a|^2 = |b|^2 d\lambda, \quad \text{and} \quad d|b|^2 = |a|^2 d\lambda.$$

Note that we can split eq. (4.57b) into real and imaginary parts as

$$d_H(e^{\lambda-\phi}\text{Re } \Phi_2) = 0, \tag{4.59a}$$

$$\text{and} \quad d_H(e^{3\lambda-\phi}\text{Im } \Phi_2) = \frac{1}{8}e^{4\lambda} \star \sigma(f). \tag{4.59b}$$

Firstly, we note that the existence of two compatible pure spinors Φ_1 and Φ_2

defines an SU(3)×SU(3) structure on $T \oplus T^*$. This is the algebraic constraint. As we saw, SU(3)×SU(3) structures elegantly generalise many of the structures on T that are of physical relevance – for instance, they encompass SU(3) and SU(2) structures.

We briefly make three remarks. Firstly, the supersymmetry conditions in eqs. (4.57a) and (4.59) are not valid in the absence of Ramond-Ramond fluxes – or rather, for $f = 0$ they do not contain information. Secondly – and we will return to this point later – the supersymmetry conditions in eqs. (4.57a) and (4.59) are only equivalent to the full equations of motion if they are supplemented with the Bianchi identities and the equations of motion of the fluxes. Thirdly, as for the SU(3) case, the intrinsic torsion components W^{ij} introduced in eq. (3.32) are entirely determined by the fluxes, the warp factor and the derivatives of the dilaton [38].

From eq. (4.57a), we see that $e^{2\lambda-\phi}\Phi_1$ satisfies the definition of a (twisted) generalised Calabi-Yau structure (à la Hitchin). Its closure under d_H, in particular, corresponds to the twisted integrability of its associated generalised almost complex structure $\mathcal{J}_1$. On the other hand, from eq. (4.59), Φ_2 fails to be closed under d_H, in fact solely due to its imaginary part, and so its associated generalised almost complex structure $\mathcal{J}_2$ is not integrable. Whilst Φ_1 and Φ_2 do form an SU(3)×SU(3) structure, as described earlier, they do not meet the integrability condition expected of a generalised Calabi-Yau *metric* structure. It is evident from eq. (4.59b) that it is precisely the Ramond-Ramond fluxes that spoil the closure of Φ_2 and therefore break the integrability of $\mathcal{J}_2$, effectively acting as a source for the Nijenhuis tensor.

There is a nice symmetry between type IIA and type IIB: in both cases, it is the pure spinor whose parity coincides with that of the Ramond-Ramond fluxes which is integrable. To gain further insight into the geometry of each type, we need to introduce a classification of pure spinors. On a 6-dimensional manifold, pure spinors can be written as

$$\Phi = e^A \wedge t_k$$

for A a complex 2-form, and t_k a holomorphic $(0 \leq k \leq 3)$-form, such that

$$A^{6-2k} \wedge t_k \wedge \bar{t}_k \neq 0.$$

The "type" of Φ is k. This is related to the generalised Darboux theorem mentioned in section 3.4, by means of which we can locally choose k holomorphic coordinates and $6-2k$ real symplectic ones, where considerations on chirality lead to k being even in type IIA and odd in type IIB. Writing $A = B - i\omega$, we notice that a $(k = 0)$-type pure spinor is (the B-transform of) $e^{-i\omega}$, and it has non-zero norm if $\omega\wedge\omega\wedge\omega \neq 0$. On the other hand, for a $(k = 3)$-type pure spinor we may take, for instance, $\Phi = \Omega$ for a complex 3-form Ω, with the non-degeneracy of the norm of the spinor translating into $\Omega \wedge \bar{\Omega} \neq 0$, and the spinor's purity implying the decomposable nature of Ω. Closure of the spinor gives $d\omega = 0$ and thus defines a symplectic geometry, as we saw. By eq. (3.31), the integrability of the resulting complex structure implies then that $d\Omega$ cannot be a (2,2)-form – in particular, if $d\Omega = 0$, the canonical bundle is holomorphically trivial – as opposed to only being topologically trivial – which we recall characterises generalised Calabi-Yau structures.

We see, then, that type IIA supergravity leads to manifolds that are (twisted) symplectic about regular points, with the exception of the case $c = 0$ (the static SU(2) structure), for which the manifold is a complex-symplectic hybrid. Type IIB is instead realised entirely on hybrid manifolds, which become purely complex[67] at points where $d = 0$. In general, then, we see that $\mathcal{N} = 1$ vacua guide us to hybrid symplectic-complex manifolds. It is clear then that generalised complex structures – which, as we remarked in chapter 3, interpolate precisely between these two geometries – provide a natural formalism to describe flux compactifications [22, 38].

Looking slightly ahead, we emphasize the appearance of the Ramond-Ramond fluxes as integrability defects for Φ_2. One may wonder whether these fluxes may be included in the twisting of the differential operator instead, as was done for non-trivial H fluxes. This would define a new notion of integrability which would apply to Φ_2 as well. This is precisely the theme of the next chapter, in which we will present how a "geometrisation" of the RR fluxes – in other words, an extension of the tangent bundle to incorporate the RR gauge transformations – can be achieved.

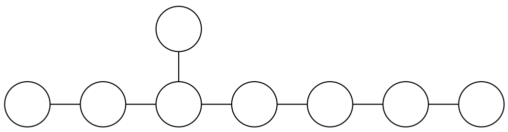
[67]Note that, unlike the symplectic structure, the complex one is not twisted [22].

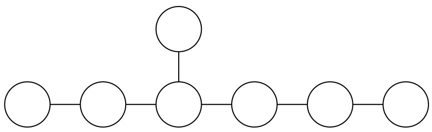
Chapter 5

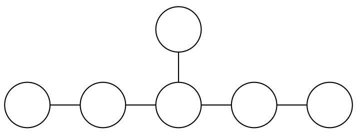
Generalising generalised geometry

5.1 Exceptional group theory

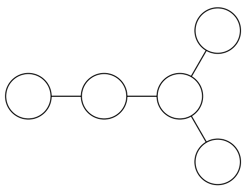
Recall that the exceptional Lie groups E_8, E_7, and E_6 correspond to the Dynkin diagrams

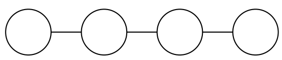
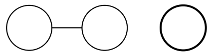
E_8  ,

E_7  ,

and E_6  .

We can define the exceptional groups E_d for $d = 5, 4, 3$ by extrapolating the Dynkin diagrams above, so that, successively deleting roots from the right,

D_5  ,

A_4  , and $\mathrm{A}_2 \oplus \mathrm{A}_1$  ,

respectively, so that we have the identifications [51]

$$\begin{aligned} \mathrm{E}_5 &\equiv \mathrm{Spin}(10), \\ \mathrm{E}_4 &\equiv \mathrm{SU}(5), \\ \text{and} \quad \mathrm{E}_3 &\equiv \mathrm{SU}(3) \times \mathrm{SU}(2). \end{aligned}$$

For these, we have the following split real forms

$$\begin{aligned} \mathrm{E}_{5(5)} &= \mathrm{Spin}(5,5), \\ \mathrm{E}_{4(4)} &= \mathrm{SL}(5;\mathbb{R}), \\ \text{and} \quad \mathrm{E}_{3(3)} &= \mathrm{SL}(3;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R}). \end{aligned}$$

5.2 U-duality and exceptional generalised geometry

It was noticed in [64] that the Kaluza-Klein n-torus compactification of eleven-dimensional supergravity produces an $(11-n)$-dimensional theory which displays a number of symmetries; in particular, the gauge groups are the exceptional groups described above, and summarised in table 5.1. Hull and Townsend conjectured in [5] that a discrete version of these groups survives as a symmetry of the full M-theory.

Geometrical formalisms which offer a natural covariance under the symmetries of a theory are very attractive for a number of reasons. Covariance under the T-duality group[68] $\mathrm{O}(d,d)$ was certainly a motivation for studying generalised geometry. It is then natural to try to extend the formalism of generalised geometry to incorporate the larger symmetry groups discussed above. In particular, we seek to extend the $\mathrm{O}(d,d)$ covariance to the $\mathrm{E}_{7(7)}$ covariance corresponding to U-duality. This extension was initially proposed by Hull [6], and Pires Pacheco and Waldram [7]. In particular, it allows us to include RR degrees of freedom within the formalism [15].

[68]Technically, the proper T-duality group is the discrete version, $\mathrm{O}(d,d;\mathbb{Z})$.

n	E_n	H_n	$\dim(\mathrm{E}_n)$	$\dim(\mathrm{E}_n/\mathrm{H}_n)$
8	$\mathrm{E}_{8(8)}$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	248	128
7	$\mathrm{E}_{7(7)}$	$\mathrm{SU}(8)/\mathbb{Z}_2$	133	70
6	$\mathrm{E}_{6(6)}$	$\mathrm{Sp}(4)/\mathbb{Z}_2$	78	42
5	$\mathrm{Spin}(5,5)$	$(\mathrm{Sp}(2)\times\mathrm{Sp}(2))/\mathbb{Z}_2$	45	25
4	$\mathrm{SL}(5;\mathbb{R})$	$\mathrm{SO}(5)$	24	14
3	$\mathrm{SL}(3;\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3)\times\mathrm{SO}(2)$	11	7
2	$\mathrm{SL}(2;\mathbb{R})\times\mathbb{R}$	$\mathrm{SO}(2)$	4	3

Table 5.1: The U-duality groups E_n, their maximal compact subgroups H_n, and the dimensionalities of E_n and E_n/H_n. Taken from [6].

5.3 Type II geometries

The perturbative charges of string theory – the momentum and winding number – can be assembled into an $\mathrm{SO}(d,d)$ vector. On the other hand, the even [odd] forms corresponding to the RR charges in type IIA [IIB] transform under the spinor representation of $\mathrm{SO}(d,d)$ [6]. This prompts us to perform the following generalisation:

$$T\oplus T^* \qquad \rightarrow \qquad T\oplus T^*\oplus S^{\pm}.$$

In fact, to accommodate the additional charges[69] in $d=6$, the appropriate generalised bundles for type IIA and type IIB are, respectively,

$$E_0 = T\oplus\Lambda^5 T\oplus\Lambda^5 T^*\oplus T^*\oplus S^{\pm}, \tag{5.1}$$

where $\Lambda^5 T$ corresponds to the (Hodge dual of the) Kaluza-Klein monopole charge, and $\Lambda^5 T^*$ to the NS fivebrane charge[70,71] [6].

The generalisation of the natural $\mathrm{O}(6,6)$ action on $T\oplus T^*$ is a natural $E_{7(7)}$ action on E_0, under which the natural symplectic form and symmetric quartic on E_0 are left invariant.

The exceptional generalised tangent bundle in eq. (5.1) amounts to a (weighted) decomposition of the fundamental representation **56** of $\mathrm{E}_{7(7)}$ under the $\mathrm{GL}(6;\mathbb{R})\subset$

[69]The generalisation to the bundle $T\oplus T^*\oplus S^{\pm}$ only suffices for $d\leq 4$ [6].
[70]Note that, in $D=11$, these charges give $\Lambda^6 T\oplus\Lambda^5 T^*$.
[71]Note that $\Lambda^5 T\oplus\Lambda^5 T^*\simeq T^*\oplus T$ for $d=6$ [6].

$E_{7(7)}$ subgroup consisting of the diffeomorphisms on the (exceptional) tangent space. To see this, we now embed $GL(6;\mathbb{R})$ into each of the two factors of $SL(2;\mathbb{R}) \times O(6,6) \subset E_{7(7)}$.

We recall that, given $M \in GL(6;\mathbb{R})$, we already defined the action of $GL(6;\mathbb{R})$ on the vector representation **12** of $O(6,6)$ in eq. (3.2). We are then left with having to embed the $GL(6;\mathbb{R})$ action into the $SL(2;\mathbb{R})$ factor. We do so by defining $M \in GL(6;\mathbb{R})$ to act on a $SL(2;\mathbb{R})$ doublet via the matrix [65]

$$\begin{pmatrix} (\det M)^{-1/2} & 0 \\ 0 & (\det M)^{1/2} \end{pmatrix}.$$

Under this decomposition, an element of **56** can be shown to transform as a section of the bundle[72] E_0 defined in eq. (5.1). We will refer to E_0 as the (untwisted) exceptional generalised tangent bundle.

Under the same $GL(6,\mathbb{R}) \subset E_{7(7)}$ subgroup, the adjoint representation **133** of $E_{7(7)}$ can be refined into

$$\mathrm{ad}\tilde{F} = (T \otimes T^*) \oplus \Lambda^2 T \oplus \Lambda^2 T^* \oplus \mathbb{R} \oplus \Lambda^6 T^* \oplus \Lambda^6 T \oplus \Lambda^- T^* \oplus \Lambda^- T. \tag{5.2}$$

To make contact with ordinary generalised geometry, we can decompose the fundamental representation **56** of $E_{7(7)}$ under the maximal subgroup $(S)O(6,6) \times SL(2;\mathbb{R})$ – whose discrete versions are the T- and S-duality symmetry groups of the full string theory [5] – as

$$\mathbf{56} = (\mathbf{12}, \mathbf{2}) + (\mathbf{32}, \mathbf{1}),$$

so that an element $\mathscr{X}$ transforming in **56** decomposes into

$$\mathscr{X} = (\mathscr{X}^{Ai}, \mathscr{X}^+),$$

where the fundamental $O(6,6)$ index $A \in \{1, \ldots, 12\}$, the $SL(2;\mathbb{R})$ index $i \in \{1,2\}$, and $\mathscr{X}^+$ is an $O(6,6)$ Weyl spinor [6, 65]. Similarly, for the adjoint representation **133** of $E_{7(7)}$,

$$\mathbf{133} = (\mathbf{1}, \mathbf{3}) + (\mathbf{66}, \mathbf{1}) + (\mathbf{32'}, \mathbf{2}),$$

[72] More precisely, elements of the fundamental representation of $E_{7(7)}$ decomposed under the $GL(6;\mathbb{R})$ subgroup transform as sections of the weighted bundle $(\Lambda^6 T^*)^{1/2} \otimes E_0$ [65].

so that, given a section μ of the adjoint bundle,

$$\mu = (\mu^i{}_j, \mu^A{}_B, \mu^{i-}), \tag{5.3}$$

where $i \in \{1,2\}$ is a $\mathrm{SL}(2,\mathbb{R})$ doublet index, and $A \in \{1,\ldots,12\}$ is an $\mathrm{O}(6,6)$ fundamental index.

5.3.1 Type IIA supergravity gauge fields

We are interested in the realisation of shifts in the fields of type IIA supergravity, namely the internal B field, the internal 6-form[73,74] $\tilde{B}$, and the formal sum $C^- = C_1 + C_3 + C_5$ of RR potentials[75]. Incorporating these shifts implies extending the framework of (ordinary) generalised geometry – which, as we saw, only includes shifts in the B field – and this is indeed possible with the generalisation to $\mathrm{E}_{7(7)}$.

We thus seek sections of the $\mathrm{E}_{7(7)}$ adjoint bundle whose exponentiation results in the gauge transformations described above; these can be identified with $B \in \Lambda^2 T^*$, $\tilde{B} \in \Lambda^6 T^*$, and $C^- \in \Lambda^- T^*$ in the decomposition in eq. (5.2), respectively. In fact, via an embedding $\mathrm{GL}(6,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(6,6) \subset \mathrm{E}_{7(7)}$, we can recognise the generator of these shifts as the element in eq. (5.3) which takes the particular form

$$(\tilde{B} v^i v_j, \left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right), v^i C^-),$$

where the $\mathrm{SL}(2,\mathbb{R})$ vector $v^i = (1,0)$, and $v_i \equiv e_{ij} v^j$ [15,65].

It can be shown that the B, $\tilde{B}$, and C^- defined above establish the following subalgebra within $\mathfrak{e}_{7(7)}$

$$[B + \tilde{B} + C^-, B' + \tilde{B}' + C^{-\prime}] = 2\langle C^-, C^{-\prime}\rangle + B \wedge C^{-\prime} - B' \wedge C^-,$$

where the 6-form $2\langle C^-, C^{-\prime}\rangle$ amounts to a $\tilde{B}$ shift, while $B \wedge C^{-\prime} - B' \wedge C^-$ to a C^- shift [15,65].

[73] This is the internal component of the 10-dimensional dual of the B field.

[74] Initially, we are only interested in shifts of the internal RR fields (and of the B field); however, in order for the gauge fields to form a closed set under the action of the U-duality group, we are forced to consider shifts in $\tilde{B}$ as well [15].

[75] Note that C^- transforms as a chiral spinor under $\mathrm{Spin}(6,6)$.

5.3.2 Exceptional twisting

Given non-zero form field strengths, only a local description of the potentials B, $\tilde{B}$, and C^- is possile. In particular, within an overlap $U_\alpha \cap U_\beta$, we must patch sections of the exceptional generalised bundle by

$$\mathscr{X}_{(\alpha)} = e^{d\lambda^+_{(\alpha\beta)}} e^{d\tilde{\lambda}_{(\alpha\beta)}} e^{d\lambda_{(\alpha\beta)}} \mathscr{X}_{(\beta)},$$

as to account for the supergravity gauge transformations [65]

$$\begin{aligned} & B_{(\alpha)} - B_{(\beta)} = d\lambda_{(\alpha\beta)}, \\ & \tilde{B}_{(\alpha)} - \tilde{B}_{(\beta)} = d\tilde{\lambda}_{(\alpha\beta)} + \langle d\lambda^+_{(\alpha\beta)}, e^{-d\lambda_{(\alpha\beta)}} C^-_{(\beta)} \rangle, \\ \text{and} \qquad & C^-_{(\alpha)} - C^-_{(\beta)} = d\lambda^+_{(\alpha\beta)} + e^{-d\lambda_{(\alpha\beta)}} C^-_{(\beta)}. \end{aligned}$$

The resulting $\mathscr{X}_{(\alpha)}$ are sections of a twisted exceptional generalised tangent bundle E, and B, $\tilde{B}$, and C^- are connections on gerbes [7].

5.3.3 SU(8)/$\mathbb{Z}_2$ structures and exceptional generalised metrics

It can be shown that the background $\{B, \tilde{B}, C^-, g, \phi\}$ parametrises $\mathrm{E}_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$, where $\mathrm{SU}(8)/\mathbb{Z}_2$ is the maximal compact subgroup of $\mathrm{E}_{7(7)}$. This is significant, because spinors transform in the fundamental representation **8** of its double cover, SU(8) [65]. The fact that the form fields, the metric, and the dilaton parametrise the above coset space is equivalent to the statement that they define an $\mathrm{SU}(8)/\mathbb{Z}_2$ structure $\mathscr{J}$ on E_0; this is an almost complex structure, i.e. $\mathscr{J}^2 = -1$, that is compatible with $\mathrm{E}_{7(7)}$, i.e.

$$\Omega(\mathscr{J}\mathscr{X}, \mathscr{J}\mathscr{Y}) = \Omega(\mathscr{X}, \mathscr{Y}) \quad \text{and} \quad q(\mathscr{J}\mathscr{X}) = q(\mathscr{X}),$$

so that $\mathscr{J}$ defines a subgroup $\mathrm{SU}(8)/\mathbb{Z}_2 \subset \mathrm{E}_{7(7)}$ [7]. In the above, Ω and q are the symplectic form and quartic invariants defining the $\mathrm{E}_{7(7)}$ structure.

In analogy with (ordinary) almost complex structures (see eq. (2.4)), the specification of an $\mathrm{SU}(8)/\mathbb{Z}_2$ structure can be recast as the partitioning of the (complexified) exceptional tangent bundle into two subbundles,

$$E \otimes \mathbb{C} = \mathcal{L} \oplus \bar{\mathcal{L}},$$

where the subbundle $\mathcal{L}$ transforms (or, more precisely, its fibres transform) in the representation[76] **28** of $\mathrm{SU}(8)/\mathbb{Z}_2$ [7].

Having equipped E with a symplectic structure Ω and an $\mathrm{SU}(8)/\mathbb{Z}_2$ structure $\mathscr{J}$, we find an exceptional, positive-definite generalised metric $\mathcal{G} \in \Gamma(S^2 E^*)$ on E,

$$\mathcal{G}(\mathscr{X}, \mathscr{X}) = \Omega(\mathscr{X}, \mathscr{J}\mathscr{X}).$$

A generic exceptional generalised metric can be constructed out of a specific one $\mathcal{G}_0$ given only by $\{g, \phi\}$ via

$$\mathcal{G}(\mathscr{X}, \mathscr{X}) = \mathcal{G}_0(e^{C^-} e^{\tilde{B}} e^{B} \mathscr{X}, e^{C^-} e^{\tilde{B}} e^{B} \mathscr{X}),$$

which is the "exceptionalisation" of eq. (4.8) [65].

5.4 M-geometries

The Type IIA geometry described in the previous section arises as the reduction of an M-theory geometry [6].

Consider a $(d{+}1)$-dimensional manifold $\mathcal{M}$ given by a circle bundle over the usual d-dimensional manifold M. We label the tangent bundles on M and $\mathcal{M}$ as T and $\mathcal{T}$, respectively, and similarly for the cotangent bundles. Projecting circle-invariant p-forms and p-vectors yields the isomorphisms [6]

$$\Lambda^p \mathcal{T}|_{\mathrm{U}(1)} \simeq \Lambda^p T_M \oplus \Lambda^{p-1} T_M \quad \text{and} \quad \Lambda^p \mathcal{T}^*|_{\mathrm{U}(1)} \simeq \Lambda^p T^*_M \oplus \Lambda^{p-1} T^*_M. \tag{5.4}$$

Consider the geometry on M given by eq. (5.1). We first note that, for $d = 6$,

$$S^+ \simeq \Lambda^+ T^* = \Lambda^0 T^* \oplus \Lambda^2 T^* \oplus \Lambda^4 T^* \oplus \Lambda^6 T^*,$$

and $\Lambda^6 T^* = \Lambda^6 T$. Therefore, for Type IIA we may write the (untwisted) generalised

[76] Note that $\mathbf{56} = \mathbf{28} + \overline{\mathbf{28}}$.

tangent bundle as

$$E_0 = T \oplus \Lambda^5 T \oplus \Lambda^5 T^* \oplus T^* \oplus \Lambda^0 T^* \oplus \Lambda^2 T^* \oplus \Lambda^4 T^* \oplus \Lambda^6 T.$$

We now note that, following the isomorphisms in eq. (5.4), the factors on the right-hand side above arise from the following projections:

$$\begin{aligned} T \oplus \Lambda^0 T^* &\simeq \mathcal{T}\big|_{\mathrm{U}(1)}\,, \\ \Lambda^2 T^* \oplus T^* &\simeq \Lambda^2 \mathcal{T}^*\big|_{\mathrm{U}(1)}\,, \\ \Lambda^5 T^* \oplus \Lambda^4 T^* &\simeq \Lambda^5 \mathcal{T}^*\big|_{\mathrm{U}(1)}\,, \\ \text{and}\quad \Lambda^6 T \oplus \Lambda^5 T &\simeq \Lambda^6 \mathcal{T}\big|_{\mathrm{U}(1)}\,. \end{aligned}$$

Therefore, we see that the type IIA geometry on M studied in the previous sections emerges as the reduction of M-geometry on $\mathcal{M}$ given by[77] [6]

$$\mathcal{E}_0 = \mathcal{T} \oplus \Lambda^2 \mathcal{T}^* \oplus \Lambda^5 \mathcal{T}^* \oplus \Lambda^6 \mathcal{T}. \tag{5.5}$$

A section $\mathscr{V} \in \Gamma(\mathcal{E}_0)$ of this bundle is then the formal sum

$$\mathscr{V} = v + \rho + \sigma + \tau, \tag{5.6}$$

where v is a vector, ρ a 2-form, σ a 5-form, and τ a 6-vector [6].

5.4.1 Geometrising the gauge symmetry of 11-dimensional supergravity

As before, we adorn the bundle with a connective structure by imposing

$$\mathscr{V}_{(\alpha)} = e^{d\lambda_{(\alpha\beta)} + d\tilde{\lambda}_{(\alpha\beta)}} \mathscr{V}_{(\beta)} \tag{5.7}$$

[77] This is often written with $\Lambda^6 \mathcal{T}$ replaced by the space of weighted one-forms, i.e. as

$$\mathcal{E}_0 = \mathcal{T} \oplus \Lambda^2 \mathcal{T}^* \oplus \Lambda^5 \mathcal{T}^* \oplus (\mathcal{T}^* \otimes \Lambda^7 \mathcal{T}^*).$$

This relies on the canonical vector bundle isomorphism $\Lambda^6 \mathcal{T} \simeq \mathcal{T}^* \otimes \Lambda^7 \mathcal{T}$, and on the further identification between $\Lambda^7 \mathcal{T}$ and $\Lambda^7 \mathcal{T}^*$ enabled by the volume form, on a 7-dimensional manifold.

on the overlap $U_{(\alpha)} \cap U_{(\beta)}$, where we take the generator $\lambda_{(\alpha\beta)}$ to be a 2-form (the reason for this should become clear by the end of this section) satisfying the consistency condition

$$\lambda_{(\alpha\beta)} + \lambda_{(\beta\gamma)} + \lambda_{(\gamma\alpha)} = d\kappa_{(\alpha\beta\gamma)}$$

on triple overlaps $U_{(\alpha)} \cap U_{(\beta)} \cap U_{(\gamma)}$, for some 1-forms $\kappa_{(\alpha\beta\gamma)}$ which in turn satisfy

$$\kappa_{(\alpha\beta\gamma)} + \kappa_{(\beta\gamma\delta)} + \kappa_{(\gamma\delta\alpha)} + \kappa_{(\delta\alpha\beta)} = g^{-1}_{(\alpha\beta\gamma\delta)} dg_{(\alpha\beta\gamma\delta)}$$

on quadruple intersections $U_{(\alpha)} \cap U_{(\beta)} \cap U_{(\gamma)} \cap U_{(\delta)}$, where, for quantised supergravity flux, the U(1)-valued functions $g_{(\alpha\beta\gamma\delta)}$ obey the cocyle condition

$$g_{(\alpha\beta\delta\gamma)} g_{(\beta\gamma\delta\epsilon)} g_{(\gamma\delta\epsilon\alpha)} g_{(\delta\epsilon\alpha\beta)} g_{(\epsilon\alpha\beta\gamma)} = 1$$

on quintuple overlaps $U_{(\alpha)} \cap U_{(\beta)} \cap U_{(\gamma)} \cap U_{(\delta)} \cap U_{(\epsilon)}$ [6,66]. The 5-form $\tilde{\lambda}_{(\alpha\beta)}$ is required to satisfy similar conditions[78], culminating in a cocycle condition on octupole overlaps.

The exponentiated action of the patching in eq. (5.7) corresponds to, at the level of the components in eq. (5.6), the transformations [66]

$$\begin{aligned}
v_{(\alpha)} - v_{(\beta)} &= 0, \\
\rho_{(\alpha)} - \rho_{(\beta)} &= \imath_{v_{(\beta)}} d\lambda_{(\alpha\beta)}, \\
\sigma_{(\alpha)} - \sigma_{(\beta)} &= d\lambda_{(\alpha\beta)} \wedge \rho_{(\beta)} + \frac{1}{2} d\lambda_{(\alpha\beta)} \wedge \imath_{v_{(\beta)}} d\lambda_{(\alpha\beta)} + \imath_{x_{(\beta)}} d\tilde{\lambda}_{(\alpha\beta)}, \\
\text{and} \quad \tau_{(\alpha)} - \tau_{(\beta)} &= i d\lambda_{(\alpha\beta)} \wedge \sigma_{(\beta)} - j d\tilde{\lambda}_{(\alpha\beta)} \wedge \rho_{(\beta)} + j d\lambda_{(\alpha\beta)} \wedge \imath_{v_{(\beta)}} d\tilde{\lambda}_{(\alpha\beta)} \\
&\quad + \frac{1}{2} j d\lambda_{(\alpha\beta)} \wedge d\lambda_{(\alpha\beta)} \wedge \rho_{(\beta)} + \frac{1}{6} j d\lambda_{(\alpha\beta)} \wedge d\lambda_{(\alpha\beta)} \wedge \imath_{v_{(\beta)}} d\lambda_{(\alpha\beta)}.
\end{aligned}$$

where we employ the "j-notation" of Pires-Pacheco and Waldram [7].

We have thus twisted the bundle $\mathcal{E}_0$ with a gerbe, to give a new bundle $\mathcal{E}$. Formally, we may define this bundle via the extensions [66]

$$\begin{aligned}
0 \longrightarrow \Lambda^2 \mathcal{T}^* \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{T} \longrightarrow 0, \\
0 \longrightarrow \Lambda^5 \mathcal{T}^* \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}'' \longrightarrow 0,
\end{aligned}$$

[78] In fact, $\lambda_{(\alpha\beta)}$ appears in the conditions for $\tilde{\lambda}_{(\alpha\beta)}$; this is a manifestation of the Chern-Simons coupling [66,67].

$$\text{and} \quad 0 \longrightarrow \mathcal{T}^*{\otimes}\Lambda^7\mathcal{T}^* \longrightarrow \quad \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0.$$

The adjoint representation **133** of $\mathrm{E}_{7(7)}$ can be expanded in terms of $\mathrm{GL}(7;\mathbb{R})$ representations as

$$\mathbf{133} = \mathbf{49} \oplus \mathbf{35} \oplus \overline{\mathbf{35}} \oplus \mathbf{7} \oplus \overline{\mathbf{7}},$$

so that the adjoint bundle decomposes into [7]

$$(\mathcal{T} \otimes \mathcal{T}^*) \oplus \Lambda^3\mathcal{T} \oplus \Lambda^3\mathcal{T}^* \oplus \Lambda^6\mathcal{T} \oplus \Lambda^6\mathcal{T}^*.$$

Hence, it is natural to include alongside the usual action of $\mathrm{GL}(7;\mathbb{R})$ that of a 3-form $A \in \Lambda^3\mathcal{T}^*$, as well as that of a 6-form $\tilde{A} \in \Lambda^6\mathcal{T}^*$. These gerbe connections are patched via [66]

$$A_{(\alpha)} - A_{(\beta)} = d\lambda_{(\alpha\beta)} \tag{5.8a}$$

$$\text{and} \quad \tilde{A}_{(\alpha)} - \tilde{A}_{(\beta)} = d\tilde{\lambda}_{(\alpha\beta)} - \frac{1}{2}d\lambda_{(\alpha\beta)} \wedge A_{(\beta)} \tag{5.8b}$$

on $U_{(\alpha)} \cap U_{(\beta)}$. This retrospectively motivates our earlier choice of $\lambda_{(\alpha\beta)}$ and $\tilde{\lambda}_{(\alpha\beta)}$ as 2- and 5-forms, respectively.

The gauge transformations in eq. (5.8) are symmetries of 11-dimensional supergravity. Indeed, by varying the Lagrangian for the bosonic sector [68],

$$\mathcal{L} = R\star 1 - \frac{1}{2}\star F \wedge F - \frac{1}{6}F \wedge F \wedge A,$$

where the field strength $F = dA$, we find the equation of motion [67]

$$d\star F + \frac{1}{2}F \wedge F = 0. \tag{5.9}$$

or equivalently[79],

$$\star F = d\tilde{A} - \frac{1}{2}A \wedge F \tag{5.10}$$

for some 6-form potential $\tilde{A}$, which is how the gerbe connection defined above

[79]Indeed, applying the exterior derivative on eq. (5.10) recovers eq. (5.9). The opposite direction consists in reformulating eq. (5.9) as the Bianchi identity $d\star F = -\frac{1}{2}d(A \wedge F)$, from which, locally at least, eq. (5.10) follows.

emerges physically. According to the doubled formalism prescription [67], we can recast (bosonic) field equations into the constraint that the original field strength be the Hodge dual of that of the newly-introduced double[80]; in this case, then, the field strength of $\tilde{A}$ must be

$$\tilde{F} = \star F = d\tilde{A} - \frac{1}{2} A \wedge F.$$

The field strengths vary as

$$\delta F = d\delta A = d^2\lambda = 0$$

and

$$\begin{aligned}
\delta\tilde{F} &= d\delta\tilde{A} - \frac{1}{2}\delta(A \wedge F) \\
&= d^2\tilde{\lambda} - \frac{1}{2} d^2\lambda \wedge A + \frac{1}{2} d\lambda \wedge F - \frac{1}{2} d\lambda \wedge F \\
&= 0
\end{aligned}$$

under the transformations $\delta A = A_{(\alpha)} - A_{(\beta)}$ and $\delta\tilde{A} = \tilde{A}_{(\alpha)} - \tilde{A}_{(\beta)}$ in eq. (5.8) (where we omitted the patch labels for clarity), so that the gauge transformations generated by closed 3-forms $d\lambda$ and closed 6-forms $d\tilde{\lambda}$ are truly symmetries of the theory.

Given the infinitesimal gauge transformations δ_Λ and $\delta_{\tilde{\Lambda}}$ by the closed 3- and 6-forms Λ and $\tilde{\Lambda}$, respectively, we find the commutators,

$$\begin{aligned}
&[\delta_\Lambda, \delta_{\tilde{\Lambda}}] = 0, \\
&[\delta_{\tilde{\Lambda}}, \delta_{\tilde{\Lambda}'}] = 0, \\
\text{and} \quad &[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{\Lambda \wedge \Lambda'},
\end{aligned}$$

meaning that two A-shifts commute to give an $\tilde{A}$-shift [67]. We see, therefore, that the gauge symmetry group of 11-dimensional supergravity is $\Omega^3_{\mathrm{cl}} \ltimes \Omega^6_{\mathrm{cl}}$, the group of closed A- and $\tilde{A}$-shifts [66].

Finally, we mention in passing that, at the level of generators, the above commutation relations result in a superalgebra, since, for instance, the generator of the

[80] Up to some subtleties involving the dilaton.

3-form potential A is fermionic [67].

5.4.2 Exceptional Courant bracket

We advance our efforts to geometrise the symmetries of 11-dimensional supergravity by defining, in analogy with the generalised geometry case, the exceptional Dorfman derivative [66]

$$\begin{aligned}\mathscr{L}_{\mathscr{V}}\mathscr{V}' &= \mathcal{L}_v v' + (\mathcal{L}_v \rho' - \imath_{v'} d\rho) + (\mathcal{L}_v \sigma' - \imath_{v'} d\sigma - \rho' \wedge d\rho) \\ &\quad + (\mathcal{L}_v \tau' - j\sigma' \wedge d\rho - j\rho' \wedge d\sigma)\end{aligned}$$

of a section $\mathscr{V}' = v' + \rho' + \sigma' + \tau' \in \Gamma(E)$ with respect to another section $\mathscr{V} = v + \rho + \sigma + \tau \in \Gamma(E)$. This new bracket incorporates both A- and $\tilde{A}$-shifts, on top of the usual diffeomorphisms.

Its antisymmetrisation gives the exceptional Courant bracket [7],

$$\begin{aligned}[\![\mathscr{V}, \mathscr{V}']\!] &= \frac{1}{2}\left(\mathscr{L}_{\mathscr{V}}\mathscr{V}' - \mathscr{L}_{\mathscr{V}'}\mathscr{V}\right) \\ &= [v, v'] + \mathcal{L}_v \omega' - \mathcal{L}_{v'}\omega - \frac{1}{2}\,\mathrm{d}\left(i_v \omega' - i_{v'}\omega\right) \\ &\quad + \mathcal{L}_v \sigma' - \mathcal{L}_{v'}\sigma - \frac{1}{2}\,\mathrm{d}\left(i_v \sigma' - i_{v'}\sigma\right) + \frac{1}{2}\omega \wedge \mathrm{d}\omega' - \frac{1}{2}\omega' \wedge \mathrm{d}\omega \\ &\quad + \frac{1}{2}\mathcal{L}_v \tau' - \frac{1}{2}\mathcal{L}_{v'}\tau + \frac{1}{2}\left(j\rho \wedge \mathrm{d}\sigma' - j\sigma' \wedge \mathrm{d}\rho\right) - \frac{1}{2}\left(j\rho' \wedge \mathrm{d}\sigma - j\sigma \wedge \mathrm{d}\rho'\right)\end{aligned}$$

whose automorphisms correspond to $\mathrm{Diff} \ltimes (\Omega^3_{\mathrm{cl}} \ltimes \Omega^6_{\mathrm{cl}})$ – precisely the local symmetry group of supergravity [66].

5.5 11-dimensional supergravity compactifications

We conclude by giving a sketch of the exceptional generalised geometry description of eleven-dimensional supergravity. Further details can be found in [7,69], on which this section is based.

We take the eleven-dimensional manifold M_{11} to be the product of some external space M_4, which we take to be Minkowski $\mathbb{R}^{3,1}$, and a compact, seven-dimensional internal manifold X_7,

$$M_{11} = M_4 \times_W X_7.$$

We consider the same metric ansatz as in eq. (4.31), except now $M, N \in \{1, \dots, 11\}$, and $m, n \in \{1, \dots, 7\}$. In analogy with eq. (4.58), to comply with Poincaré invariance we must take a field strength F of the form

$$F = \mathcal{F} + \star_7 \tilde{\mathcal{F}} \wedge \mathrm{vol}_4,$$

where $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are 4- and 7-forms on the internal manifold. This split provokes a decomposition of the equations of motion (eq. (5.9)) and the Bianchi identity which recovers the gerbe structure of the gauge fields and therefore the patching structure given in eq. (5.8).

Following the compactification ansatz, we can decompose an eleven-dimensional spinor ϵ into [7]

$$\epsilon = \xi_+ \otimes \eta + \xi_- \otimes \eta,$$

for a complex Spin(7) spinor η. This transforms in $\mathbf{8}$, the fundamental representation of the SU(8) double cover of the SU(8)/$\mathbb{Z}_2$ structure defined by the exceptional generalised metric $\mathcal{G}$.

The internal spinor is globally non-vanishing, and so it reduces the structure group of $\mathcal{E}$ from SU(8) down to SU(7). This is in analogy with our earlier discussions on generalised geometry; similarly, the supersymmetry conditions associated with flux compactifications can be described in terms of integrable SU(7) structures [69]. Thus, we see that the geometries of supersymmetric flux backgrounds are the "exceptionalisations" (as in, the exceptional analogues) of the complex structures familiar from conventional geometry.

We now ask if the description of the SU(7) structure in terms of $(\mathcal{G}, \eta)$ can be rephrased in terms of a single element in some representation of $E_{7(7)}$.

The fundamental representation, i.e. that of the exceptional generalised bundle, decomposes under U(1) $\times$ SU(7) into four eigenbundles,

$$\mathbf{7_3} \oplus \mathbf{21_{-1}} \oplus \overline{\mathbf{21}}_{\mathbf{1}} \oplus \overline{\mathbf{7}}_{\mathbf{-3}},$$

where the subscript labels the U(1) charge. We see there are no SU(7) singlets. The adjoint representation expands into

$$\mathbf{1_0} \oplus \mathbf{48_0} \oplus \mathbf{35_2} \oplus \overline{\mathbf{35}}_{\mathbf{-2}} \oplus \overline{\mathbf{7}}_{\mathbf{-4}} \oplus \overline{\mathbf{7}}_{\mathbf{4}}.$$

We note the appearance of an SU(7) singlet – however, this is also a singlet under U(1), so that it actually gives rise to an $\mathbb{R}^+ \times \mathrm{U}(7)$ structure. We proceed with decomposing representations of increasing dimensionality, by turning to the 912-dimensional representation often labelled $\tilde{K}$. It can be shown that

$$\tilde{K}_{\mathbb{C}} \sim \mathbf{1_7} \oplus \mathbf{7_3} \oplus \overline{\mathbf{35}}_{\mathbf{5}} \oplus \mathbf{140_3} \oplus \overline{\mathbf{224}}_{\mathbf{1}} \oplus \overline{\mathbf{28}}_{\mathbf{1}} \oplus \overline{\mathbf{21}}_{\mathbf{1}} \oplus \text{c.c.},$$

so that we finally witness the appearance of a true SU(7) singlet. By studying the transformation properties of a complex element ϕ of this singlet, we can confirm that it is indeed stabilised by SU(7). We see that the existence of a globally non-zero tensor which transforms in $\mathrm{E}_{7(7)}$'s **912** representation defines an SU(7) structure on the exceptional generalised tangent space $\mathcal{E}$ [7].

Chapter 6

Conclusion

In this dissertation, we attempted to give a brief overview of the mathematical framework of generalised geometry, and to describe some of its applications in string theory.

The nature of this work inevitably called for a review of several facets of ordinary complex differential geometry. In particular, we emphasised the importance of G-structures and their related integrability. Furthermore, we introduced several geometries and provided a description of them in terms of invariant tensors, and therefore of G-structures – for instance, we characterised a Calabi-Yau geometry in terms of the vanishing of the intrinsic torsion of an SU(3) structure.

We described the extension of the tangent bundle T over a manifold to $T \oplus T^*$. We saw that this naturally came with an $\mathrm{O}(d,d)$-invariant metric, and then introduced a generalised Riemannian metric to break down the structure group further to its maximal compact subgroup $\mathrm{O}(d) \times \mathrm{O}(d)$. Generalised complex structures were then introduced and shown to attractively interpolate between complex and symplectic geometries. In order to define the integrability of such structures, we generalised the Lie bracket and found the Dorfman and Courant brackets. Anticipating the need to describe fermionic degrees of freedom, we took the lift to $\mathrm{Spin}(d,d)$, and introduced generalised spinors. We showed the existence of a correspondence between differential polyforms and $\mathrm{Clifford}(d,d)$ spinors, and between (lines of) pure spinors and generalised almost complex structures. These associations allowed us to recast the integrability of generalised structures into differential conditions on spinors, and thus define certain generalised geometries, including the Calabi-Yau case, solely in terms of pure spinors. We ended the mathematical part of the thesis by describing how $T \oplus T^*$ could be twisted with a gerbe. This enlarged the structure group of the bundle to the geometric subgroup $\mathrm{GL}(d; \mathbb{R}) \ltimes \Omega^2_{\mathrm{cl}}$.

The overarching theme of the dissertation, however, was the application of generalised geometry to type II superstring theory, and in particular to its low-energy limit, type II supergravity. We gradually embedded the bosonic content of the theory into the formalism of generalised geometry. We saw how the metric and Kalb-Ramond enter the framework in the form of a generalised metric, thus appearing on equal footing. The inclusion of the dilaton in terms of the isomorphism between generalised spinors and polyforms was more subtle. The gauge symmetry of the theory was captured by twisting the generalised tangent bundle itself, so that its transition functions featured the B-gauge transformations alongside the usual diffeomorphisms. Alternatively, we found that the twisting could appear explicitly at the level of the integrability structures.

T-duality was presented as one of the main points of contact with generalised geometry. The perturbative charges of string theory were shown to form an $\mathrm{SO}(d, d)$ vector; the isometry required by a T-duality via the Buscher procedure was shown to be naturally described in terms of a section of the generalised tangent bundle, which could be interpreted as a generalised Killing vector for the generalised metric.

We then turned to compactifications of type II supergravity. The further assumption of supersymmetry greatly simplified the equations of motion. We derived the conditions imposed by supersymmetry on the geometries of the two manifolds: namely, in the case of fluxless compactifications, a Minkowski external space fibred by a Calabi-Yau internal space. We remarked the geometrical nature of the deformations in the effective theory, before moving on to consider the case with Neveu-Schwarz-Neveu-Schwarz and Ramond-Ramond fluxes. We found that these flux compactifications admitted an elegant and concise description in terms of generalised geometries, and specifically (twisted) generalised Calabi-Yau metric structures, with the Ramond-Ramond fluxes acting as integrability defects for one of the pure spinors.

It was precisely the nature of the Ramond-Ramond fluxes as integrability defects that motivated the discussion of exceptional generalised geometry. This was presented as an extension of the framework of generalised geometry aimed at geometrising the action of the U-duality group. We described the form of the twisting of the tangent bundle, as well the exceptionalisation of the structures of generalised geometry. Finally, we sketched the description of 11-dimensional supergravity flux compactifications in terms of exceptional complex structures and their integrability.

Generalised geometry is a relatively recent area of physics, and so it is also an active field of research. Current efforts include extending the formalism of exceptional generalised geometry to encompass AdS flux backgrounds, and using the AdS/CFT correspondence to probe superconformal field theories [70].

We also mention a field that is closely related to generalised geometry, namely that of double field theory, devised by Hull and Zwiebach [71–73]. This formalism builds on string field theory by introducing dual coordinates associated with the winding modes. The doubling of spacetime itself naturally leads to a doubling of the corresponding tangent space, and could therefore provide a clearer physical interpretation of generalised geometry.

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